**Chapter 3** **Curve Sketching**

**Chapter 3 Prerequisite Skills**

**Chapter 3 Prerequisite Skills** **Question 1 Page 148**

**a)** Let $f(x) = x^3 + 2x^2 + 2x + 1$.

$f(1) = 6$

$f(-1) = 0$ $(x+1)$ is a factor.

$x^3 + 2x^2 + 2x + 1 = (x+1)(x^2 + x + 1)$

**b)** Let $f(z) = z^3 - 6z - 4$.

$f(1) = -9$

$f(-1) = 1$

$f(2) = -8$

$f(-2) = 0$ $(z+2)$ is a factor.

$z^3 - 6z + 4 = (z+2)(z^2 - 2z - 2)$

**c)** Let $f(t) = t^3 + 6t^2 - 7t - 60$.

$f(1) = -60$

$f(-1) = -48$

$f(2) = -42$

$f(-2) = -30$

$f(3) = 0$ $(t-3)$ is a factor.

$$\begin{aligned} t^3 + 6t^2 - 7t - 60 &= (t-3)(t^2 + 9t + 20) \\ &= (t-3)(t+5)(t+4) \end{aligned}$$

**d)** Let $f(b) = b^3 + 8b^2 + 19b + 12$.

$f(1) = 40$

$f(-1) = 0$ $(b+1)$ is a factor.

$$\begin{aligned} b^3 + 8b^2 + 19b + 12 &= (b+1)(b^2 + 7b + 12) \\ &= (b+1)(b+4)(b+3) \end{aligned}$$

**e)**
$$\begin{aligned} 3n^3 - n^2 - 3n + 1 &= n^2(3n-1) - (3n-1) \\ &= (3n-1)(n^2-1) \\ &= (3n-1)(n+1)(n-1) \end{aligned}$$

**f)** Let $f(p) = 2p^3 - 9p^2 + 10p - 3$.

$f(1) = 0$ $\quad$ $(p-1)$ is a factor.

$$\begin{aligned} 2p^3 - 9p^2 + 10p - 3 &= (p-1)(2p^2 - 7p + 3) \\ &= (p-1)(2p-1)(p-3) \end{aligned}$$

**g)**
$$\begin{aligned} 4k^3 + 3k^2 - 4k - 3 &= k^2(4k+3) - (4k+3) \\ &= (4k+3)(k^2-1) \\ &= (4k+3)(k+1)(k-1) \end{aligned}$$

**h)** Let $f(w) = 6w^3 - 11w^2 - 26w + 15$.

$f(1) = -16$

$f(-1) = 24$

$f(3) = 0$ $\quad$ $(w-3)$ is a factor.

$$\begin{aligned} 6w^3 - 11w^2 - 26w + 15 &= (w-3)(6w^2 + 7w - 5) \\ &= (w-3)(6w^2 + 10w - 3w - 5) \\ &= (w-3)(2w(3w+5) - (3w+5)) \\ &= (w-3)(3w+5)(2w-1) \end{aligned}$$

**Chapter 3 Prerequisite Skills** $\qquad$ **Question 2 Page 148**

**a)**
$$\begin{aligned} x^2 - 7x + 12 &= 0 \\ (x-3)(x-4) &= 0 \\ x = 3, x &= 4 \end{aligned}$$

**b)**
$$\begin{aligned} 4x^2 - 9 &= 0 \\ (2x+3)(2x-3) &= 0 \\ x = -\frac{3}{2}, x &= \frac{3}{2} \end{aligned}$$

**c)**
$$\begin{aligned} 18v^2 &= 36v \\ 18v^2 - 36v &= 0 \\ 18v(v-2) &= 0 \\ v &= 0,\ 2 \end{aligned}$$

**d)**
$$\begin{aligned} a^2 + 5a &= 3a + 35 \\ a^2 + 2a - 35 &= 0 \\ (a+7)(a-5) &= 0 \\ a = -7,\ a &= 5 \end{aligned}$$

**e)** $4.9t^2 - 19.6t + 2.5 = 0$

$49t^2 - 196t + 25 = 0$

$$t = \frac{196 \pm \sqrt{(-196)^2 - 4(49)(25)}}{98}$$

$$t = \frac{196 \pm \sqrt{33516}}{98}$$

$$t = \frac{98 \pm \sqrt{8379}}{49}$$

$t \approx 3.87,\ t \approx 0.132$

**f)** $x^3 + 6x^2 + 3x - 10 = 0$

$(x-1)(x^2 + 7x + 10) = 0$ Use the factor theorem.

$(x-1)(x+2)(x+5) = 0$

$x = 1,\ x = -2,\ x = -5$

**g)** $\dfrac{x^2 - 5x - 14}{x^2 - 1} = 0$ $x \neq \pm 1$

$x^2 - 5x - 14 = 0$

$(x-7)(x+2) = 0$

$x = 7,\ x = -2$

**Chapter 3 Prerequisite Skills** **Question 3 Page 148**

**a)** $2x - 10 > 0$

$2x > 10$ Add 10.

$x > 5$ Multiply by $\frac{1}{2}$.

**b)** $x(x+5) < 0$

The roots of the related equation, $x(x+5) = 0$, are $x = 0$ and $x = -5$.

Test the inequality for arbitrary values in the intervals $(-\infty, -5)$, $(-5, 0)$, and $(0, \infty)$.

For the first interval, use $x = -8$.

L.S. $= -8(-8+5)$

$= 24$

$\not< 0$

For the second interval, use $x = -1$.

$$\begin{aligned} \text{L.S.} &= -1(-1+5) \\ &= -4 \\ &< 0 \end{aligned}$$

For the third interval, use $x = 1$.

$$\begin{aligned} \text{L.S.} &= 1(1+5) \\ &= 5 \\ &\not< 0 \end{aligned}$$

Since only $x = -1$ satisfies the inequality, the solution is $-5 < x < 0$.

**c)** $x^2(x-4) > 0$

The roots of the related equation, $x^2(x-4) = 0$, are $x = 0$ and $x = 4$.
Test the inequality for arbitrary values in the intervals $(-\infty, 0), (0, 4)$, and $(4, \infty)$.

For the first interval, use $x = -1$.

$$\begin{aligned} \text{L.S.} &= (-1)^2(\text{Ğ}1\ \text{Ğ}4) \\ &= -5 \\ &\not> 0 \end{aligned}$$

For the second interval, use $x = 1$.

$$\begin{aligned} \text{L.S.} &= 1^2(1-4) \\ &= -3 \\ &\not> 0 \end{aligned}$$

For the third interval, use $x = 5$.

$$\begin{aligned} \text{L.S.} &= (5)^2(5-4) \\ &= 25 \\ &> 0 \end{aligned}$$

Since only $x = 5$ satisfies the inequality, the solution is $x > 4$.

**d)** $x^2 + 5x - 14 < 0$

Solve the related equation.

$$\begin{aligned} x^2 + 5x - 14 &= 0 \\ (x+7)(x-2) &= 0 \\ x = -7,\ x &= 2 \end{aligned}$$

The roots of the related equation, $x^2 + 5x - 14 = 0$ are $x = -7$ and $x = 2$.

Test the inequality for arbitrary values in the intervals $(-\infty, -7), (-7, 2)$, and $(2, \infty)$.

For the first interval, use $x = -10$.

$$\begin{aligned} \text{L.S.} &= (-10)^2 + 5(-10) - 14 \\ &= 36 \\ &\not< 0 \end{aligned}$$

For the second interval, use $x = 0$.

$$\begin{aligned} \text{L.S.} &= 0^2 + 5(0) - 14 \\ &= -14 \\ &< 0 \end{aligned}$$

For the third interval, use $x = 3$.

$$\begin{aligned} \text{L.S.} &= (3)^2 + 5(3) - 14 \\ &= 10 \\ &\not< 0 \end{aligned}$$

Since only $x = 0$ satisfies the inequality, the solution is $-7 < x < 2$.

**e)** $(x-3)(x+2)(x-1) > 0$

The roots of the related equation, $(x-3)(x+2)(x-1) = 0$ are $x = -2$, $x = 1$, and $x = 3$.

Test the inequality for arbitrary values in the intervals $(-\infty, -2), (-2, 1), (1, 3)$, and $(3, \infty)$.

For the first interval, use $x = -3$.

$$\begin{aligned} \text{L.S.} &= (-3-3)(-3-3)(-3+2)(-3-1) \\ &= -24 \\ &\not> 0 \end{aligned}$$

For the second interval, use $x = 0$.

$$\begin{aligned} \text{L.S.} &= (0-3)(0-3)(0+2)(0-1) \\ &= 6 \\ &> 0 \end{aligned}$$

For the third interval, use $x = 2$.

$$\begin{aligned} \text{L.S.} &= (2-3)(2+2)(2-1) \\ &= -4 \\ &\not> 0 \end{aligned}$$

For the fourth interval, use $x = 4$.

$$\begin{aligned} \text{L.S.} &= (4-3)(4+2)(4-1) \\ &= 18 \\ &> 0 \end{aligned}$$

Since both $x = 0$ and $x = 4$ satisfy the inequality, the solution is $-2 < x < 1$ or $x > 3$.

**f)** The expression changes sign when either the numerator or the denominator is zero.
The possible values for change of sign are $x = 0$, $x = 1$, and $x = -1$.

Test the inequality for arbitrary values in the intervals $(-\infty, -1), (-1, 0)$, $(0, 1)$, and $(1, \infty)$.

For the first interval, use $x = -2$

$$\begin{aligned} \text{L.S.} &= \frac{-2}{(-2)^2 - 1} \\ &= -\frac{2}{3} \\ &\not> 0 \end{aligned}$$

For the second interval, use $x = -0.5$.

$$\begin{aligned} \text{L.S.} &= \frac{-0.5}{(-0.5)^2 - 1} \\ &= \frac{2}{3} \\ &> 0 \end{aligned}$$

For the third interval, use $x = 0.5$.

$$\begin{aligned} \text{L.S.} &= \frac{0.5}{(0.5)^2 - 1} \\ &= -\frac{2}{3} \\ &\not> 0 \end{aligned}$$

For the fourth interval, use $x = 2$.

$$\begin{aligned} \text{L.S.} &= \frac{2}{(2)^2 - 1} \\ &= \frac{2}{3} \\ &> 0 \end{aligned}$$

Since both $x = -0.5$ and $x = 2$ satisfy the inequality, the solution is $-1 < x < 0$ or $x > 1$.
Restriction: The restriction on $x$ occurs when the denominator is zero. Therefore, $x \neq \pm 1$.

**Chapter 3 Prerequisite Skills** **Question 4 Page 148**

**a)** To find the $x$-intercepts, let $f(x) = 0$.

$$\begin{aligned} 5x - 15 &= 0 \\ x &= 3 \end{aligned}$$

The $x$-intercept is 3.

**b)** To find the $x$-intercepts, let $g(x)=0$.

$$x^2-3x-28=0$$
$$(x-7)(x+4)=0$$

$$x=7,\ x=-4$$

The $x$-intercepts are –4 and 7.

**c)** To find the $x$-intercepts, let $h(x)=0$.

$$x^3+6x^2+11x+6=0$$

$$h(-1)=(-1)^3+6(-1)^2+11(-1)+6 \qquad \text{Use the factor theorem.}$$
$$=0 \qquad (x+1) \text{ is a factor.}$$

$$(x+1)(x^2+5x+6)=0$$
$$(x+1)(x+2)(x+3)=0$$

$$x=-1,\ x=-2, x=-3$$

The $x$-intercepts are –1, –2, and –3.

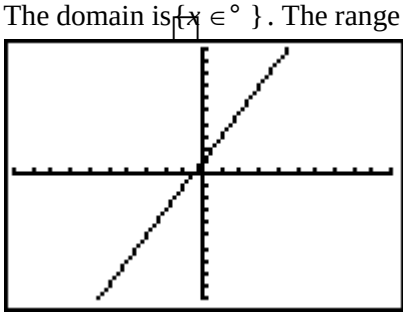
**d)** To find the $x$-intercepts, let $y$= 0.

$$\frac{x^2-9}{x^2+1}=0$$
$$x^2-9=0$$
$$(x+3)(x-3)=0$$
$$x=-3, x=3$$

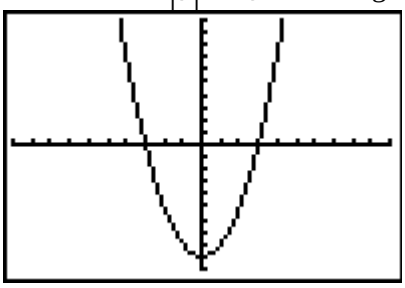
The $x$-intercepts are –3 and 3.

**Chapter 3 Prerequisite Skills** **Question 5 Page 148**

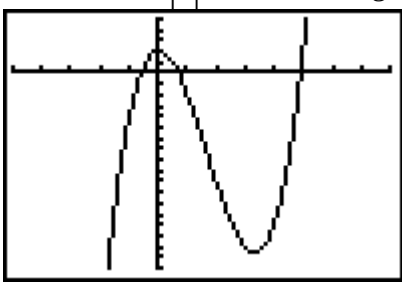
**a)** This is a line, not parallel to either axis.
The domain is $\{x \in \mathbb{R}\}$. The range is $\{y \in \mathbb{R}\}$.



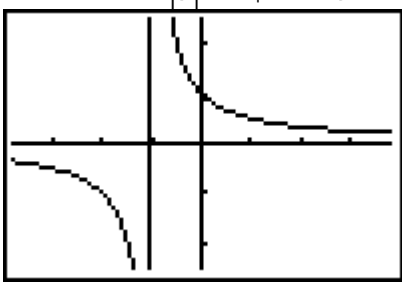
**b)** This is a parabola with vertex at (0, –9) and opening up.
The domain is $\{x \in \mathbb{R}\}$. The range is $\{y \in \mathbb{R} \mid y \geq -9\}$.



**c)** This is a cubic function.
The domain is $\{x \in \mathbb{R}\}$. The range is $\{y \in \mathbb{R}\}$.



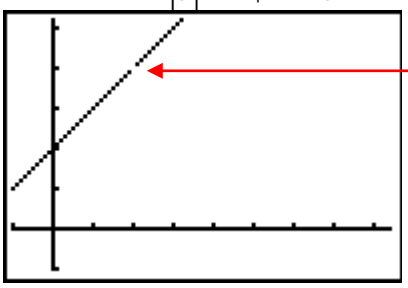
**d)** This is a rational function with vertical asymptote $x = -1$ and horizontal asymptote $y = 0$.
The domain is $\{x \in \mathbb{R} \mid x \neq -1\}$. The range is $\{y \in \mathbb{R} \mid y \neq 0\}$.



**e)** This function is linear since

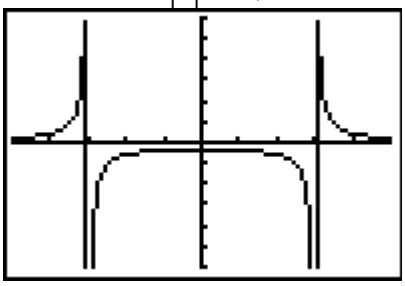
$$\frac{x^2 - 4}{x - 2} = \frac{(x+2)(x-2)}{x-2} \qquad (x \neq 2)$$
$$= x + 2$$

The domain is $\{x \in \mathbb{R} \mid x \neq 2\}$. The range is $\{y \in \mathbb{R} \mid y \neq 4\}$.

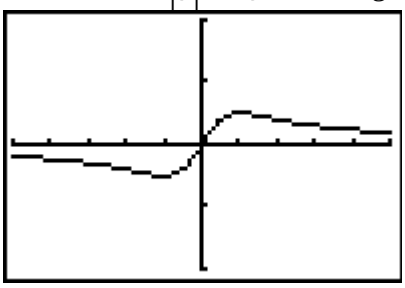


The value of $y$ does not exist at $x = 2$.
This is indicated by a break in the graph.

**f)** This is a rational function with vertical asymptotes $x = -3$ and $x = 3$ and horizontal asymptote $y = 0$.
The domain is $\{x \in \mathbb{R} \mid x \neq \pm 3\}$. The range is $\{y \in \mathbb{R} \mid y \neq 0\}$.



**g)** This is a rational function without vertical asymptotes since $x^2 + 1 \neq 0$ for any values of $x$.
A graph is required to determine the domain and range.
The domain is $\{x \in \mathbb{R}\}$. The range is $\{y \in \mathbb{R} \mid -0.5 \leq y \leq 0.5\}$.



**Chapter 3 Prerequisite Skills** **Question 6 Page 148**

See question 5 above.

**a)** None

**b)** None

**c)** None

**d)** $x = 1$, $y = 0$

**e)** None

**f)** $x = 3$, $x = -3$, $y = 0$

**g)** $y = 0$

**Chapter 3 Prerequisite Skills** **Question 7 Page 148**

**a)** The function is increasing on the intervals $(-\infty, 0)$ and $(2, \infty)$.
The function is decreasing on the interval $(0, 2)$.

**b)** The function is increasing on the intervals $(-2,0)$ and $(2,\infty)$.

The function is decreasing on the intervals $(-\infty,-2)$ and $(0,2)$.

**Chapter 3 Prerequisite Skills** **Question 8 Page 149**

**a)** $f(x)=5x^2-7x+12$

$f'(x)=10x-7$

**b)** $y=x^3-2x^2+4x-8$

$\frac{dy}{dx}=3x^2-4x+4$

**c)** $f(x)=\frac{1}{x}$

$=x^{-1}$

$f'(x)=(-1)x^{-2}$

$=-\frac{1}{x^2}$

**d)** $y=\frac{x^2-9}{x^2+1}$

$=(x^2-9)(x^2+1)^{-1}$

$\frac{dy}{dx}=(x^2-9)(-1)(x^2+1)^{-2}(2x)+2x(x^2+1)^{-1}$

$=(x^2+1)^{-2}\left[2x(-x^2+9)+2x(x^2+1)\right]$

$=(x^2+1)^{-2}\left[-2x^3+18x+2x^3+2x\right]$

$=\frac{20x}{(x^2+1)^2}$

**Chapter 3 Prerequisite Skills** **Question 9 Page 149**

$V(x)=lwh$

$=(60-2x)(40-2x)(x)$

$=4x^3-200x^2+2400x$

There is a restriction on the possible values for $x$ since its value must be positive and cannot exceed half the value of the width of the sheet of tin; i.e., $0<x<20$.

**Chapter 3 Prerequisite Skills** **Question 10 Page 149**

$V = \pi r^2 h$

$= 1000$ ①

S.A. $= 2\pi r^2 + 2\pi rh$ ②

You want a formula for S.A., only in terms of $r$, so eliminate the variable $h$.

Substitute $h = \dfrac{1000}{\pi r^2}$ from ① in formula ②

$$\text{S.A.} = 2\pi r^2 + 2\pi r\left(\frac{1000}{\pi r^2}\right)$$

$$= 2\pi r^2 + \frac{2000}{r}$$

**Chapter 3 Prerequisite Skills** **Question 11 Page 149**

**a)** Since this graph is symmetric about the origin, the function is odd.

**b)** Since this graph is symmetric about the $y$-axis, the function is even.

**c)** Since this graph is symmetric about the $y$-axis, the function is even.

**d)** Since this graph is neither symmetric about the $y$-axis nor the origin, the function is neither even nor odd.

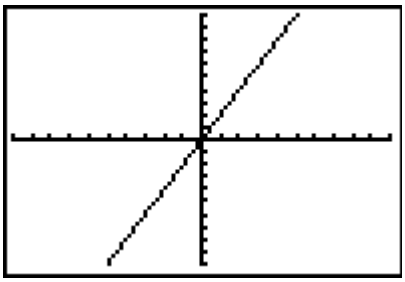
**Chapter 3 Prerequisite Skills** **Question 12 Page 149**

**a)** $f(-x) = 2(-x)$

$= -2x$

$= -f(x)$

This is an odd function.



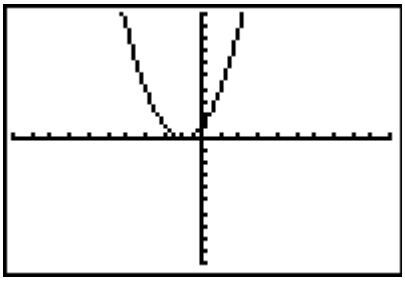
**b)** $r(-x) = (-x)^2 + 2(-x) + 1$

$= x^2 - 2x + 1$

$r(x) = x^2 + 2x + 1$

$-r(x) = -x^2 - 2x - 1$

The function is neither even nor odd.

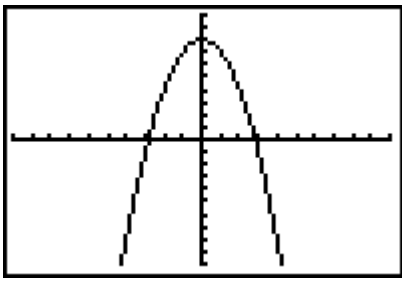


**c)** $f(-x) = -(-x)^2 + 8$

$= -x^2 + 8$

$= f(x)$

This is an even function.



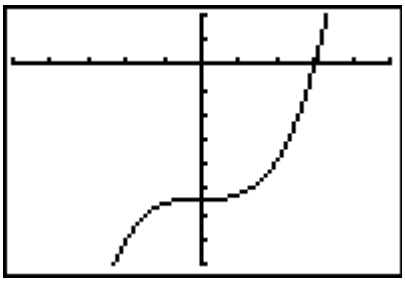
**d)** $s(-t) = (-t)^3 - 27$

$= -t^3 - 27$

$s(t) = t^3 - 27$

$-s(t) = -t^3 + 27$

The function is neither even nor odd.



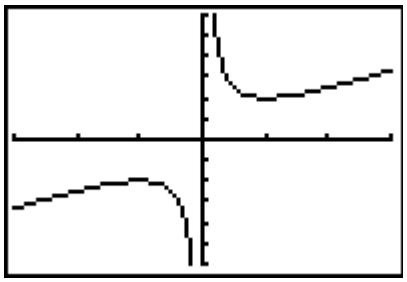
(Yscl = 5)

**e)** $h(\check{G}x) = (\check{G}x) + \dfrac{1}{(\check{G}x)}$

$$= -x - \frac{1}{x}$$

$$= -h(x)$$

This is an odd function.

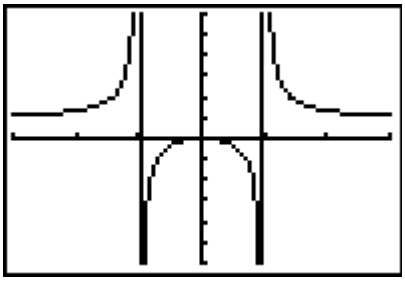


**f)** $f(\check{G}x) = \dfrac{(\check{G}x)^2}{(\check{G}x)^2 - 1}$

$$= \frac{x^2}{x^2 - 1}$$

$$= f(x)$$

This is an even function.



**Chapter 3 Section 1** **Increasing and Decreasing Functions**

**Chapter 3 Section 1** **Question 1 Page 156**

**a)** $15-5x=0$

$$-5x=-15$$
$$x=3$$

**b)** $x^2+8x-9=0$

$$(x+9)(x-1)=0$$

$$x=-9,\ x=1$$

**c)** $3x^2-12=0$

$$3x^2=12$$
$$x^2=4$$
$$x=\pm 2$$

**d)** $x^3-6x^2=0$

$$x^2(x-6)=0$$

$$x=0,\ x=6$$

**e)** $x^2+2x-4=0$

$$x=\frac{-2\pm\sqrt{(2)^2-4(1)(\check{G}4)}}{2(1)}$$ Use the quadratic formula.

$$x=\frac{-2\pm\sqrt{20}}{2}$$ $\sqrt{20}=2\sqrt{5}$

$$x=-1\pm\sqrt{5}$$

**f)** $x^3-3x^2-18x+40=0$

$$f(1)=1-3-18+40=20$$
$$\neq 0$$
$$f(-1)=-1-3+18+40$$
$$=54$$
$$\neq 0$$
$$f(2)=8-12-36+40$$
$$=0$$ $(x-2)$ is a factor.

$(x-2)(x^2-x-20)=0$ Use the factor theorem.

$$(x-2)(x\ \check{G}5)(x+4)=0$$
$$x=2,\ x=5,\ x=-4$$

**g)** $x^3+3x^2-4x-12=0$

$$x^2(x+3)-4(x+3)=0$$
$$(x+3)(x^2-4)=0$$
$$(x+3)(x+2)(x-2)=0$$

$$x=-3, x=-2, x=2$$

**h)** $x^4-x^3-x^2+x=0$

$$x^3(x-1)-x(x-1)=0$$
$$(x-1)(x^3-x)=0$$
$$(x-1)(x)(x^2-1)=0$$
$$x(x-1)(x+1)(x-1)=0$$

$$x=0,\ x=1,\ x=-1$$

**Chapter 3 Section 1 Question 2 Page 156**

**a)** The function *f* is increasing when $f'(x)=15-5x>0$.
Test the inequality for arbitrary values in the intervals $(-\infty, 3)$ and $(3, \infty)$.

For the first interval, use $x = 0$.

$$\begin{aligned}\text{L.S.}&=15-5(0)\\&=15\\&>0\end{aligned}$$

For the second interval, use $x = 4$.

$$\begin{aligned}\text{L.S.}&=15-5(4)\\&=-5\\&\not>0\end{aligned}$$

*f* is increasing on the interval $(-\infty, 3)$.
*f* is decreasing on the interval $(3, \infty)$.

**b)** The function *h* is increasing when $h'(x)=x^2+8x-9>0$.

Test the inequality for arbitrary values in the intervals $(-\infty, -9), (-9, 1)$, and $(1, \infty)$.
For the first interval, use $x = -10$.

$$\begin{aligned}\text{L.S.}&=(-10)^2+8(-10)-9\\&=11\\&>0\end{aligned}$$

For the second interval, use $x = 0$.

$$\begin{aligned} \text{L.S.} &= (0)^2 + 8(0) - 9 \\ &= -9 \\ &\not> 0 \end{aligned}$$

For the third interval, use $x = 2$.

$$\begin{aligned} \text{L.S.} &= (2)^2 + 8(2) - 9 \\ &= 11 \\ &> 0 \end{aligned}$$

$h$ is increasing on the intervals $(-\infty, -9)$ and $(1, \infty)$.
$h$ is decreasing on the interval $(-9, 1)$.

**c)** The function $g$ is increasing when $g'(x) = 3x^2 - 12 > 0$.

Test the inequality for arbitrary values in the intervals $(-\infty, -2)$, $(-2, 2)$, and $(2, \infty)$.
For the first interval, use $x = -3$.

$$\begin{aligned} \text{L.S.} &= 3(-3)^2 - 12 \\ &= 15 \\ &> 0 \end{aligned}$$

For the second interval, use $x = 0$.

$$\begin{aligned} \text{L.S.} &= 3(0)^2 - 12 \\ &= -12 \\ &\not> 0 \end{aligned}$$

For the third interval, use $x = 3$.

$$\begin{aligned} \text{L.S.} &= 3(3)^2 - 12 \\ &= 15 \\ &> 0 \end{aligned}$$

$g$ is increasing on the intervals $(-\infty, -2)$ and $(2, \infty)$.
$g$ is decreasing on the interval $(-2, 2)$.

**d)** The function $f$ is increasing when $f'(x) = x^3 - 6x^2 > 0$.

Test the inequality for arbitrary values in the intervals $(-\infty, 0), (0, 6)$, and $(6, \infty)$.
For the first interval, use $x = -3$.

$$\begin{aligned} \text{L.S.} &= (-3)^3 - 6(-3)^2 \\ &= -81 \\ &\not> 0 \end{aligned}$$

For the second interval, use $x = 1$.

$$\begin{aligned} \text{L.S.} &= (1)^3 - 6(1)^2 \\ &= -5 \\ &\not> 0 \end{aligned}$$

For the third interval, use $x = 7$.

$$\begin{aligned} \text{L.S.} &= (7)^3 - 6(7)^2 \\ &= 49 \\ &> 0 \end{aligned}$$

$f$ is increasing on the interval $(6, \infty)$.
$f$ is decreasing on the intervals $(-\infty, 0)$ and $(0, 6)$.

**e)** The function $d$ is increasing when $d'(x) = x^2 + 2x - 4 > 0$.

Test the inequality for arbitrary values in the intervals $\left(-\infty, -1-\sqrt{5}\right)\left(-1-\sqrt{5}, -1+\sqrt{5}\right)$ and $\left(-1+\sqrt{5}, \infty\right)$.

For the first interval, use $x = -4$.

$$\begin{aligned} \text{L.S.} &= (-4)^2 + 2(-4) - 4 \\ &= 4 \\ &> 0 \end{aligned}$$

For the second interval, use $x = 0$.

$$\begin{aligned} \text{L.S.} &= (0)^2 + 2(0) - 4 \\ &= -4 \\ &\not> 0 \end{aligned}$$

For the third interval, use $x = 2$.

$$\begin{aligned} \text{L.S.} &= (2)^2 + 2(2) - 4 \\ &= 4 \\ &> 0 \end{aligned}$$

$d$ is increasing on the intervals $\left(-\infty, -1-\sqrt{5}\right)$ and $\left(-1+\sqrt{5}, \infty\right)$.
$d$ is decreasing on the interval $\left(-1-\sqrt{5}, -1+\sqrt{5}\right)$.

**f)** The function $k$ is increasing when $k'(x) = x^3 - 3x^2 - 18x + 40 > 0$.

Test the inequality for arbitrary values in the intervals $(-\infty, -4)$, (Ğ4, 2), (2, 5), and $(5, \infty)$.
For the first interval, use $x = -5$.

$$\begin{aligned} \text{L.S.} &= (-5)^3 - 3(-5)^2 - 18(-5) + 40 \\ &= -70 \\ &\not> 0 \end{aligned}$$

For the second interval, use $x = 0$.

$$\begin{aligned} \text{L.S.} &= (0)^3 - 3(0)^2 - 18(0) + 40 \\ &= 40 \\ &> 0 \end{aligned}$$

For the third interval, use $x = 3$.

$$\begin{aligned} \text{L.S.} &= (3)^3 - 3(3)^2 - 18(3) + 40 \\ &= -14 \\ &\not> 0 \end{aligned}$$

For the fourth interval, use $x = 6$.

$$\begin{aligned} \text{L.S.} &= (6)^3 - 3(6)^2 - 18(6) + 40 \\ &= 40 \\ &> 0 \end{aligned}$$

$k$ is increasing on the intervals $(-4, 2)$ and $(5, \infty)$.
$k$ is decreasing on the intervals $(-\infty, -4)$ and $(2, 5)$.

**g)** The function $b$ is increasing when $b'(x0 = x^3 + 3x^2 - 4x - 12 > 0$.

Test the inequality for arbitrary values in the intervals $(-\infty, -3)$, $(-3, -2)$, $(-2, 2)$, and $(2, \infty)$.
For the first interval, use $x = -4$.

$$\begin{aligned} \text{L.S.} &= (-4)^3 + 3(-4)^2 - 4(-4) - 12 \\ &= -12 \\ &\not> 0 \end{aligned}$$

For the second interval, use $x = -2.5$.

$$\begin{aligned} \text{L.S.} &= (-2.5)^3 + 3(-2.5)^2 - 4(-2.5) - 12 \\ &= 1.125 \\ &> 0 \end{aligned}$$

For the third interval, use $x = 0$.

$$\begin{aligned} \text{L.S.} &= (0)^3 + 3(0)^2 - 4(0) - 12 \\ &= -12 \\ &\not> 0 \end{aligned}$$

For the fourth interval, use $x = 3$.

$$\begin{aligned} \text{L.S.} &= (3)^3 + 3(3)^2 - 4(3) - 12 \\ &= 30 \\ &> 0 \end{aligned}$$

$b$ is increasing on the intervals $(-3, -2)$ and $(2, \infty)$.
$b$ is decreasing on the intervals $(-\infty, -3)$ and $(-2, 2)$.

**h)** The function $f$ is increasing when $f'(x) = x^4 - x^3 - x^2 + x > 0$.

Test the inequality for arbitrary values in the intervals $(-\infty, -1), (-1, 0)$, $(0,1)$, and $(1, \infty)$.
For the first interval, use $x = -2$.

$$\begin{aligned} \text{L.S.} &= (-2)^4 - (-2)^3 - (-2)^2 + (-2) \\ &= 18 \\ &> 0 \end{aligned}$$

For the second interval, use $x = -0.5$.

$$\begin{aligned} \text{L.S.} &= (-0.5)^4 - (-0.5)^3 - (-0.5)^2 + (-0.5) \\ &= -0.5625 \\ &\not> 0 \end{aligned}$$

For the third interval, use $x = 0.5$.

$$\begin{aligned} \text{L.S.} &= (0.5)^4 - (0.5)^3 - (0.5)^2 + (0.5) \\ &= 0.1875 \\ &> 0 \end{aligned}$$

For the fourth interval, use $x = 2$.

$$\begin{aligned} \text{L.S.} &= (2)^4 - (2)^3 - (2)^2 + (2) \\ &= 6 \\ &> 0 \end{aligned}$$

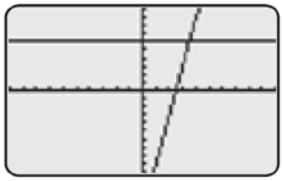
$f$ is increasing on the intervals $(-\infty, -1)$, $(0, 1)$, and $(1, \infty)$.
$f$ is decreasing on the interval $(-1, 0)$.

**Chapter 3 Section 1 Question 3 Page 156**

**a) i)** $f(x) = 6x - 15$
$f'(x) = 6$

**ii)**

**iv)**



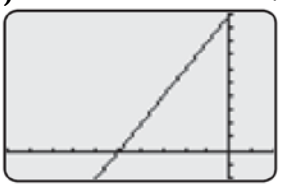
**iii)** $f$ is always increasing.
$f$ is never decreasing.

**b) i)** $f(x) = (x+5)^2$

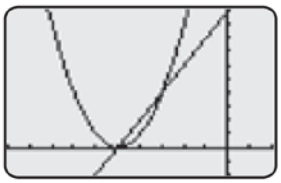
$= x^2 + 10x + 25$

$f'(x) = 2x + 10$

**ii)**



**iv)**

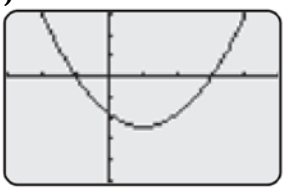


**iii)** $f$ is increasing on the interval $(-5, \infty)$.
$f$ is decreasing on the interval $(-\infty, -5)$.

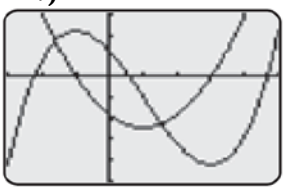
**c) i)** $f(x) = x^3 - 3x^2 - 9x + 6$

$f'(x) = 3x^2 - 6x - 9$

**ii)**



**iv)**



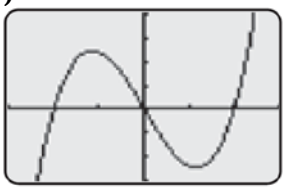
**iii)** $f$ is increasing on the intervals $(-\infty, -1)$ and $(3, \infty)$.
$f$ is decreasing on the interval $(-1, 3)$.

**d) i)** $f(x) = (x^2 - 4)^2$

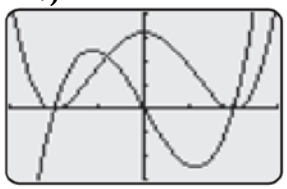
$= x^4 - 8x^2 + 16$

$f'(x) = 4x^3 - 16x$

**ii)**



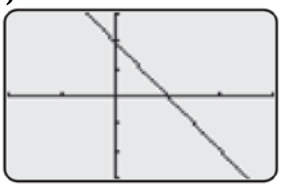
**iv)**



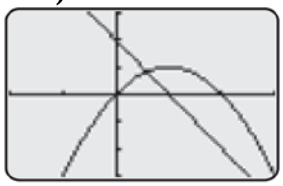
**iii)** $f$ is increasing on the intervals $(-2, 0)$ and $(2, \infty)$.
$f$ is decreasing on the intervals $(-\infty, -2)$ and $(0, 2)$.

**e) i)** $f(x) = 2x - x^2$
$f'(x) = 2 - 2x$

**ii)**



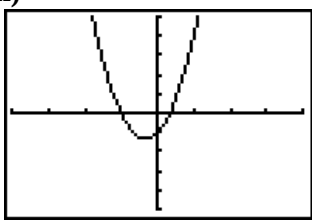
**iv)**



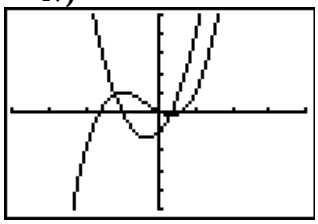
**iii)** $f$ is increasing on the interval $(-\infty, 1)$.
$f$ is decreasing on the interval $(1, \infty)$.

**f) i)** $f(x) = x^3 + x^2 - x$
$f'(x) = 3x^2 + 2x - 1$

**ii)**



**iv)**

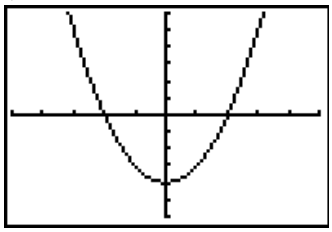


**iii)** $f$ is increasing on the intervals $(-\infty, -1)$ and $\left(\frac{1}{3}, \infty\right)$.

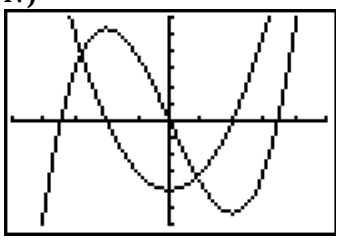
$f$ is decreasing on the interval $\left(-1, \frac{1}{3}\right)$.

**g) i)** $f(x) = \frac{1}{3}x^3 - 4x$
$f'(x) = x^2 - 4$

**ii)**



**iv)**

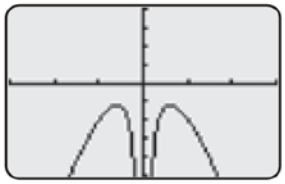


**iii)** $f$ is increasing on the intervals $(-\infty, -2)$ and $(2, \infty)$.
$f$ is decreasing on the interval $(-2, 2)$.

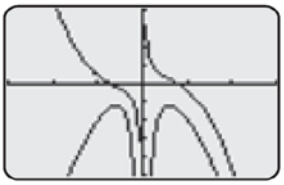
**h) i)**

$$\begin{aligned} f(x) &= \frac{1}{x} - 3x^3 \\ &= x^{-1} - 3x^3 \\ f'(x) &= -1x^{-2} - 9x^2 \\ &= -\frac{1}{x^2} - 9x^2 \end{aligned}$$

**ii)**



**iv)**



**iii)** $f$ is always decreasing (except for $x = 0$ when the function is not defined).

**Chapter 3 Section 1 Question 4 Page 156**

**a)** $f$ is increasing on the interval $(3, \infty)$.
$f$ is decreasing on the interval $(-\infty, 3)$.

**b)** $f$ is increasing on the interval $(-\infty, 8)$.
$f$ is decreasing on the interval $(8, \infty)$.

**c)** $f$ is increasing on the intervals $(-\infty, -1)$ and $(2, \infty)$.
$f$ is decreasing on the interval $(-1, 2)$.

**d)** $f$ is increasing on the interval $(-2, 2)$.
$f$ is decreasing on the intervals $(-\infty, -2)$ and $(2, \infty)$.

**e)** $f$ is increasing on the intervals $(-\infty, -5)$ and $(1, \infty)$.
$f$ is decreasing on the interval $(-5, 1)$.

**f)** $f$ is increasing on the intervals $(-2, 0)$ and $(0, \infty)$.
$f$ is decreasing on the interval $(-\infty, -2)$.

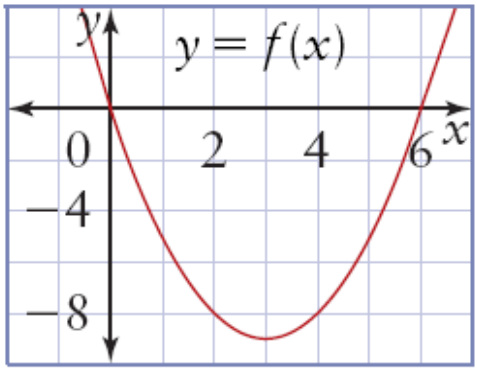
**g)** $f$ is increasing on the intervals $(-4, 0)$ and $(2, \infty)$.
$f$ is decreasing on the intervals $(-\infty, -4)$ and $(0, 2)$.

**h)** $f$ is increasing on the interval $(1, \infty)$.
$f$ is decreasing on the interval $(-\infty, 1)$.

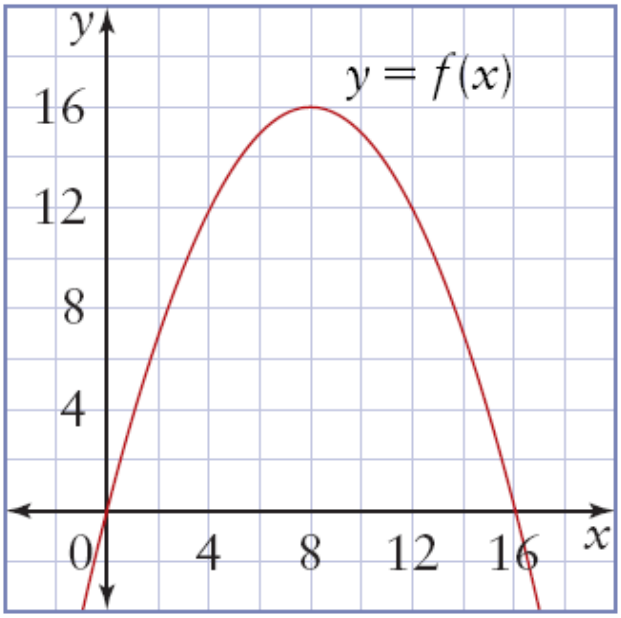
**Chapter 3 Section 1 Question 5 Page 156**

Many function graphs are possible. They need only satisfy the conditions for intervals of increase and decrease from question 4.

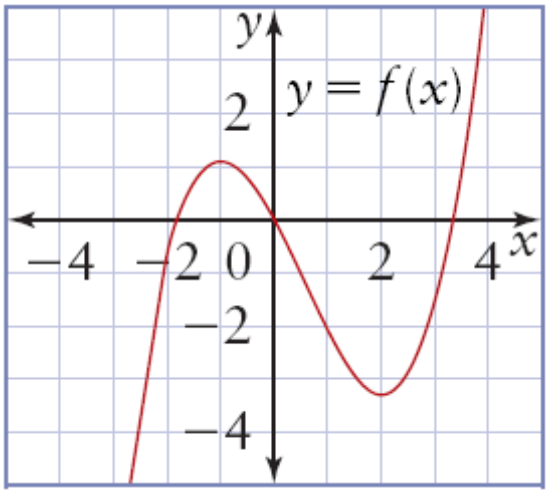
**a)**



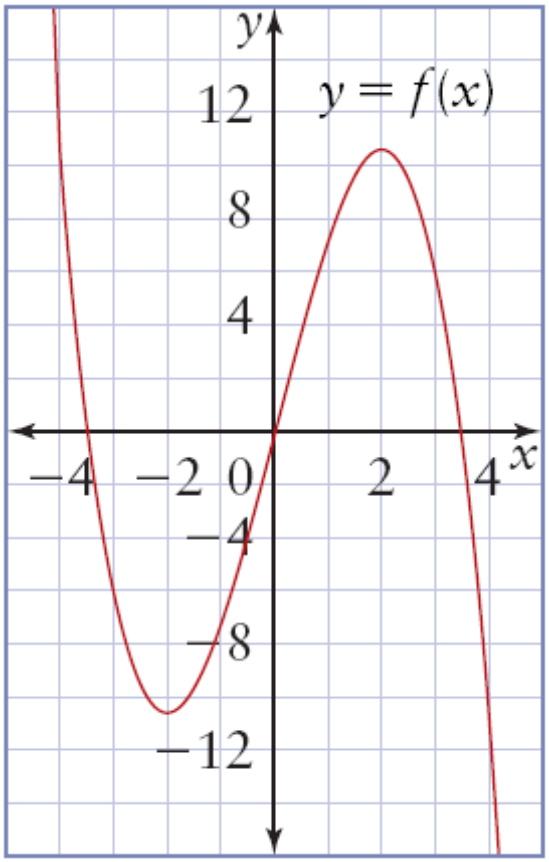

**b)**



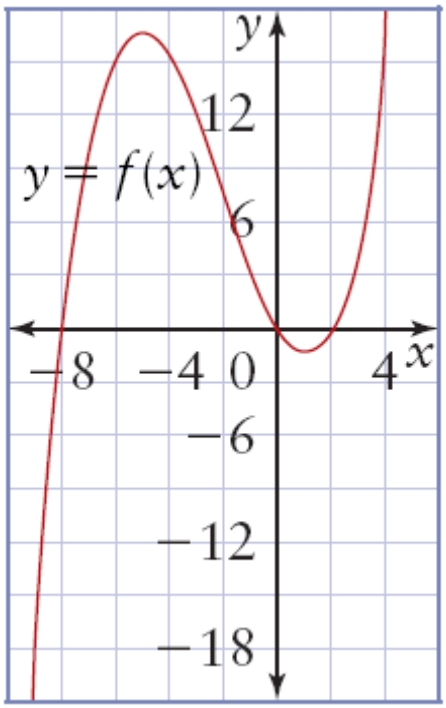
**c)**



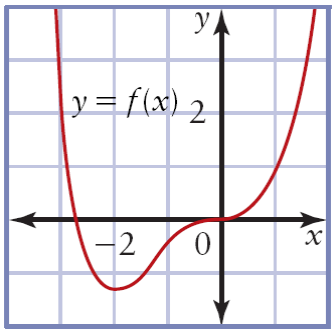

**d)**



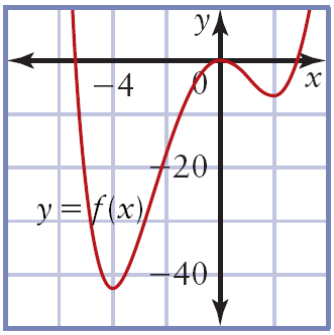
**e)**



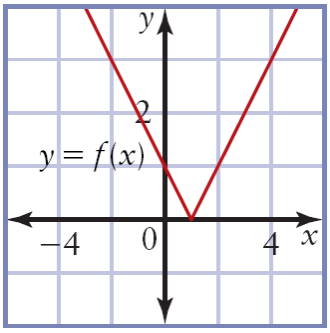

**f)**




**g)**



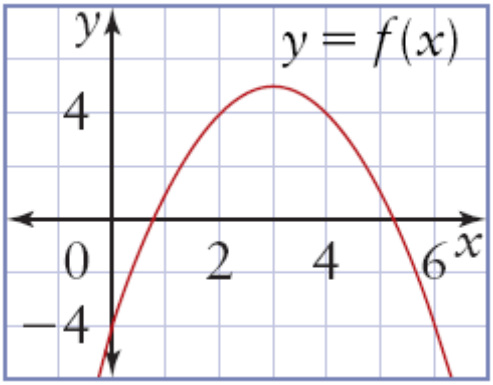
**h)** Note that there is no derivative at $x = 1$.




**Chapter 3 Section 1 Question 6 Page 157**

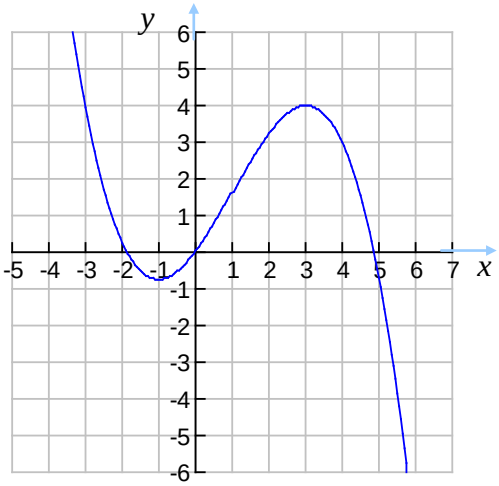
Many function graphs are possible. They need only satisfy the conditions for intervals of increase (when $f'(x) > 0$) and decrease (when $f'(x) < 0$) and pass through the specified points given.

**a)**

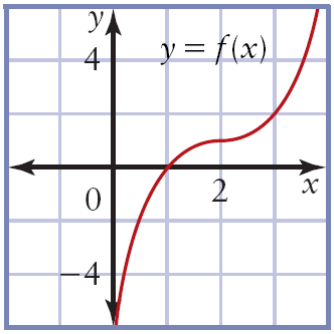



**b)** <Graph is not quite correct and needs to be revised in text answers.

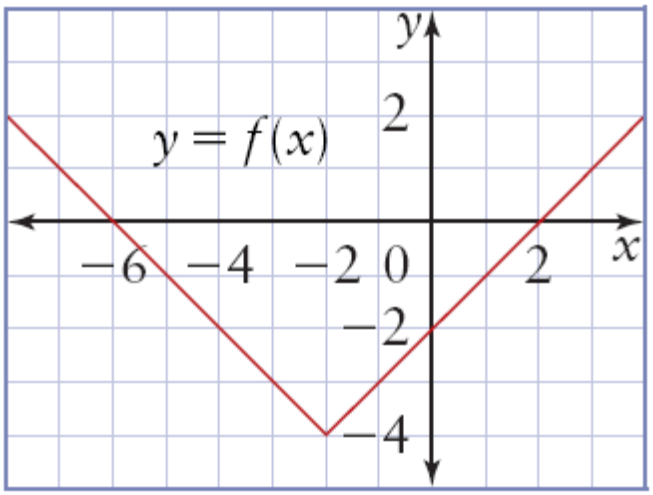
Use $f(x) = \left(-\frac{4}{27}\right)(x^3 - 3x^2 - 9x)$ as shown.



**c)**




**d)**




**Chapter 3 Section 1 Question 7 Page 157**

**a)** $k(x)$ is decreasing in the interval (0, 6) since $k'(x) < 0$ in this interval. The values $x = 3$ and $x = 5$ are both in this interval. Therefore $k(3) > k(5)$.

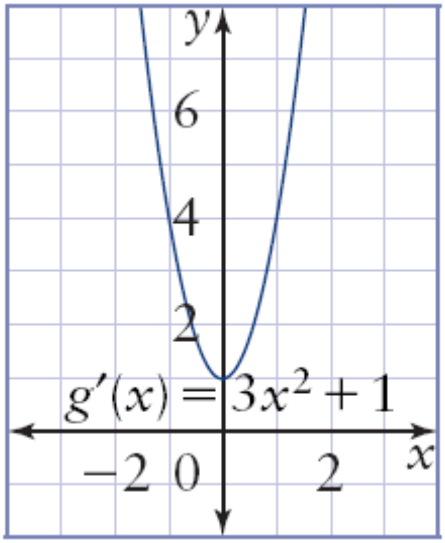
**b)** $k(x)$ is increasing in the interval $(6, \infty)$ since $k'(x) > 0$ in this interval. The values $x = 8$ and $x = 12$ are both in this interval. Therefore $k(12) > k(8)$.

**c)** $k(x)$ is decreasing from $x$ = 5 to $x$ = 6 and increasing from $x$ = 6 to $x$ = 9. Also, the (average) rate of increase (from 6 to 9) is greater than the average rate of decrease (from 5 to 6). Therefore $k(9) > k(5)$.

**d)** $k(x)$ is increasing at both $x$ = –2 and $x$ = 10. Between $x$ = –2 and $x$ = 10, $k(x)$ is increasing for 6 units along the $x$-axis and decreasing for 6 units as well. However, the average rate of increase is greater than the average rate of decrease in these intervals. Therefore $k(10) > k(-2)$.

**Chapter 3 Section 1 Question 8 Page 157**

**a)** Draw the graph of $y = g'(x)$ or $3x^2 + 1$.




The graph of $g'(x)$ is always above the $x$-axis and so is always positive. This implies the function $g(x)$ is always increasing.

**b)** Use the rules for manipulating inequalities.

$x^2 > 0$ For all $x \in \circ$ .

$3x^2 > 0$ Multiply boths sides of the inequality by 3.

$3x^2 + 1 > 0$ Add 1 to both sides of the inequality.

Therefore, $g'(x) = 3x^2 + 1 > 0$. For all $x \in \circ$ .

**Chapter 3 Section 1 Question 9 Page 157**

**a)** $h(x) = (x^2 + 2x - 3) + (x + 5)$

$= x^2 + 3x + 2$

$h'(x) = 2x + 3$

The function is increasing when $h'(x) = 2x + 3 > 0$.

First solve the related equation.

$2x + 3 = 0$

$x = -\frac{3}{2}$

Test the inequality for arbitrary values in the intervals $(-\infty, -1.5)$ and $(-1.5, \infty)$.

For the first interval, use $x = -2$.

$\text{L.S.} = 2(-2) + 3$

$= -1$

$\not> 0$

For the second interval, use $x = 0$.

$$\begin{aligned} \text{L.S.} &= 2(0)+3 \\ &= 3 \\ &> 0 \end{aligned}$$

$h$ is increasing on the interval $(-1.5, \infty)$.
$h$ is decreasing on the interval $(-\infty, -1.5)$.

**b)** $$\begin{aligned} h(x) &= f(g(x)) = (x+5)^2 + 2(x+5) - 3 \\ &= x^2 + 12x + 32 \\ h'(x) &= 2x + 12 \end{aligned}$$

The function is increasing when $h'(x) = 2x + 12 > 0$.

First solve the related equation.

$$\begin{aligned} 2x + 12 &= 0 \\ x &= -6 \end{aligned}$$

Test the inequality for arbitrary values in the intervals $(-\infty, -6)$ and $(-6, \infty)$.
For the first interval, use $x = -7$.

$$\begin{aligned} \text{L.S.} &= 2(-7)+12 \\ &= -2 \not> 0 \end{aligned}$$

For the second interval, use $x = 0$.

$$\begin{aligned} \text{L.S.} &= 2(0)+12 \\ &= 12 \\ &> 0 \end{aligned}$$

$h$ is increasing on the interval $(-6, \infty)$.
$h$ is decreasing on the interval $(-\infty, -6)$.

**c)** $$\begin{aligned} h(x) &= f(x) - g(x) + 2 \\ &= (x^2 + 2x - 3) - (x+5) + 2 \\ &= x^2 + x - 6 \\ h'(x) &= 2x + 1 \end{aligned}$$

The function is increasing when $h'(x) = 2x + 1 > 0$.

First solve the related equation.

$$\begin{aligned} 2x + 1 &= 0 \\ x &= -0.5 \end{aligned}$$

Test the inequality for arbitrary values in the intervals $(-\infty, -0.5)$ and $(-0.5, \infty)$.

For the first interval, use $x = -1$.

$$\begin{aligned} \text{L.S.} &= 2(-1)+1 \\ &= -1 \\ &\not> 0 \end{aligned}$$

For the second interval, use $x = 0$.

$$\begin{aligned} \text{L.S.} &= 2(0)+1 \\ &= 1 \\ &> 0 \end{aligned}$$

$h$ is increasing on the interval $(-0.5, \infty)$.
$h$ is decreasing on the interval $(-\infty, -0.5)$.

**d)** $$\begin{aligned} h(x) &= f(x) \times g(x) \\ &= (x^2+2x-3)(x+5) \\ &= x^3+7x^2+7x-15 \end{aligned}$$

$$h'(x) = 3x^2+14x+7$$

The function is increasing when $h'(x) = 3x^2+14x+7 > 0$.

First solve the related equation.

$$3x^2+14x+7=0$$

$$x = \frac{-14 \pm \sqrt{14^2-84}}{6}$$

$$x = \frac{-14 \pm \sqrt{112}}{6}$$

$$x = \frac{-14 \pm 4\sqrt{7}}{6}$$

$$x = \frac{-7 \pm 2\sqrt{7}}{3}$$

$$x \approx -0.6, \; x \approx -4.1$$

Test the inequality for arbitrary values in the intervals
$\left(-\infty, \frac{-7-2\sqrt{7}}{3}\right)$, $\left(\frac{-7-2\sqrt{7}}{3}, \frac{-7+2\sqrt{7}}{3}\right)$, and $\left(\frac{-7+2\sqrt{7}}{3}, \infty\right)$.

For the first interval, use $x = -5$.

$$\begin{aligned} \text{L.S.} &= 3(-5)^2+14(-5)+7 \\ &= 12 \\ &> 0 \end{aligned}$$

For the second interval, use $x = -1$.

$$\begin{aligned} \text{L.S.} &= 3(-1)^2 + 14(-1) + 7 \\ &= -4 \\ &\not> 0 \end{aligned}$$

For the third interval, use $x = 0$.

$$\begin{aligned} \text{L.S.} &= 3(0)^2 + 14(0) + 7 \\ &= 7 \\ &> 0 \end{aligned}$$

$h$ is increasing on the intervals $\left(-\infty, \frac{-7-2\sqrt{7}}{3}\right)$ and $\left(\frac{-7+2\sqrt{7}}{3}, \infty\right)$.

$h$ is decreasing on the interval $\left(\frac{-7-2\sqrt{7}}{3}, \frac{-7+2\sqrt{7}}{3}\right)$.

**Chapter 3 Section 1 Question 10 Page 157**

**a)** The function is increasing when $f'(x) = x(x-1)(x+2) > 0$.

First solve the related equation.

$$x(x-1)(x+2) = 0$$

$$x = 0,\ x = 1,\ x = -2$$

Test the inequality for arbitrary values in the intervals $(-\infty, -2)$, $(-2, 0)$, $(0, 1)$, and $(1, \infty)$.

For the first interval, use $x = -3$.

$$\begin{aligned} \text{L.S.} &= -3(-3-1)(-3+2) \\ &= -12 \\ &\not> 0 \end{aligned}$$

For the second interval, use $x = -1$.

$$\begin{aligned} \text{L.S.} &= -1(-1-1)(-1+2) \\ &= 2 \\ &> 0 \end{aligned}$$

For the third interval, use $x = 0.5$.

$$\begin{aligned} \text{L.S.} &= 0.5(0.5-1)(0.5+2) \\ &= -0.625 \\ &\not> 0 \end{aligned}$$

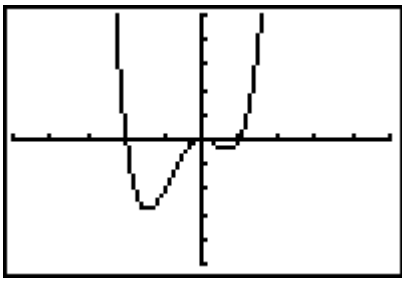
For the fourth interval, use $x = 2$.

$$\begin{aligned} \text{L.S.} &= 2(2-1)(2+2) \\ &= 8 \\ &> 0 \end{aligned}$$

$f$ is increasing on the intervals $(-2, 0)$ and $(1, \infty)$.
$f$ is decreasing on the intervals $(-\infty, -2)$ and $(0,1)$.

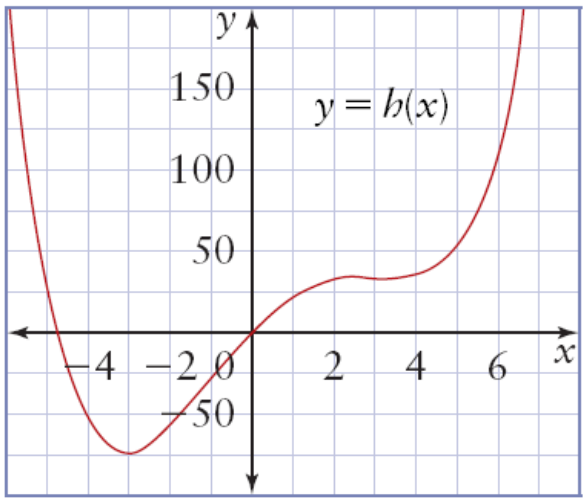
**b)** The additional factor in the derivative polynomial would alter the pattern of alternating increasing and decreasing intervals. The (positive) quartic derivative polynomial will start with a decreasing interval. Typically the intervals alternate between increasing and decreasing but the appearance of a repeated root ($x = 0$) alters the pattern. You can imagine an additional interval (0, 0) in the sequence of alternating intervals. The previous sequence of –, +, –, + becomes +, –, +, –, + where the second '+' belongs to the (imaginary) interval (0, 0).

$f$ is increasing on the intervals $(-\infty, -2)$ and $(1, \infty)$.
$f$ is decreasing on the intervals $(-2, 0)$ and $(0, 1)$.
This can be confirmed by technology. Below is the graph of $f'(x)$.



**Chapter 3 Section 1** **Question 11 Page 157**

**a)** A possible sketch is shown.




**b)** The sketch above can be translated vertically without changing the intervals of increase or decrease.

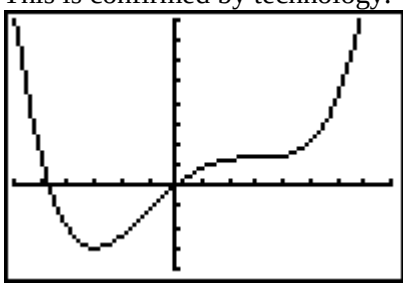
**c)** Since the function has turning points at –3 and 3, the derivative could have the form $h'(x) = (x+3)(x-3)^2$. Note the inclusion of a repeated factor since the curve does not change direction at $x = 3$.

$$h'(x) = (x+3)(x-3)^2$$
$$= (x+3)(x^2 - 6x + 9)$$
$$= x^3 - 3x^2 - 9x + 27$$

What function could have this as its derivative?

$h(x) = \frac{1}{4}x^4 - x^3 - \frac{9}{2}x^2 + 27x$ would be a possible function.

This is confirmed by technology.



$(-11 < y < 200,\ \text{Yscl} = 25)$

**Chapter 3 Section 1 Question 12 Page 157**

**a)** $A(0) = \left(\frac{4+\pi}{4\pi}\right)(0)^2 - 10(0) + 100$

$= 100$

This represents the area of the gardens if the quarter circle is not used. The 20 m of edging could be used totally for the two sides of the square garden. Each side would be 10 m and the resulting area would be 100 $m^2$. Therefore, an answer of 100 is sensible.

**b)** $A'(x) = 2\left(\frac{4+\pi}{4\pi}\right)x - 10$

$= \left(\frac{4+\pi}{2\pi}\right)x - 10$

The function is increasing when $A'(x) = \left(\frac{4+\pi}{2\pi}\right)x - 10 > 0$.

First solve the related equation.

$\left(\frac{4+\pi}{2\pi}\right)x - 10 = 0$

$x = 10\left(\frac{2\pi}{4+\pi}\right)$

$x = \frac{20\pi}{4+\pi}$

$x$ B 8.8

Test the inequality for arbitrary values in the intervals $\left(5, \frac{20^1}{4+^1}\right)$ and $\left(\frac{20^1}{4+^1}, 15\right)$.

Note the restrictions on $x$. Since each piece of edging must be at least 5 m long, $x \in [5, 15]$.

For the first interval, use $x = 5$.

$$\text{L.S.} = \left(\frac{4+^1}{2^1}\right)(5) - 10$$

B –4.3

$\sqcap$ $\not> 0$

For the second interval, use $x = 10$.

$$\text{L.S.} = \left(\frac{4+^1}{2^1}\right)(10) - 10$$

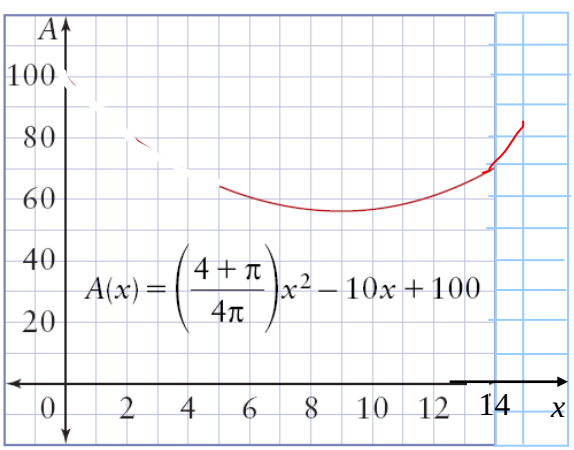
B 1.4

$\sqcap$ $> 0$

$A$ is increasing on the interval $\left(\frac{20^1}{4+^1}, 15\right)$.

$A$ is decreasing on the interval $\left(5, \frac{20^1}{4+^1}\right)$.

**c)** The graph clearly shows that the curve changes from decreasing to increasing at about $x = 9$.



**Chapter 3 Section 1 Question 13 Page 157**

**a)** $n(t) = 100 + 32t^2 - t^4$

$n'(t) = 64t - 4t^3$

The function is increasing when $n'(x) = 64t - 4t^3 > 0$.

First solve the related equation.

$$64t - 4t^3 = 0$$
$$4t(16 - t^2) = 0$$
$$4t(4 + t)(4 - t) = 0$$

$t = -4, t = 0, t = 4$

However, note that the domain is restricted to $0 < t < 5$.

Test the inequality for arbitrary values in the intervals $(0, 4)$ and $(4, 5)$.

For the first interval, use $x = 1$.

$$\begin{aligned} \text{L.S.} &= 64(1) - 4(1)^3 \\ &= 60 \\ &> 0 \end{aligned}$$

For the second interval, use $x = 5$.

$$\begin{aligned} \text{L.S.} &= 64(5) - 4(5)^3 \\ &= -180 \\ &\not> 0 \end{aligned}$$

The results can be summarized in a table.

| | $t = 0$ | $0 < t < 4$ | $t = 4$ | $t > 4$ |
|---|---|---|---|---|
| $n'(t)$ | 0 | Positive | 0 | Negative |
| $n(t)$ | Horizontal | Increasing | Horizontal | Decreasing |

**b)** If it were possible to have $t$-values for $t < 0$, the population would be increasing for $x < -4$.
The function would be horizontal at $x = -4$.
The function would be decreasing for $-4 < t < 0$.
There would be no change after $t = 5$; the function would continue to decrease.

**Chapter 3 Section 1 Question 14 Page 158**

**a)** $R(s) = -\frac{1}{2000}s^2 + 2s - 1200$

$$R(2100) = -\frac{1}{2000}(2100)^2 + 2(2100) - 1200$$
$$= 795$$

The range of the aircraft is 795 mi. when the engine speed is 2100 rpm.

**b)** $R'(s) = -\frac{1}{2000}2s + 2$

$$= -\frac{1}{1000}s + 2$$

The function is increasing when $R'(s) = -\frac{1}{1000}s + 2 > 0$.

First solve the related equation.

$$-\frac{1}{1000}s + 2 = 0$$
$$s = 2000$$

Test the inequality for arbitrary values in the intervals $(1000, 2000)$ and $(2000, 3100)$.
Note the restrictions on *s*. $(1000 < s < 3100)$
For the first interval, use $x = 1500$.

$$\text{L.S.} = -\frac{1}{1000}(1500) + 2$$
$$= 0.5$$
$$> 0$$

For the second interval, use $x = 3000$.

$$\text{L.S.} = -\frac{1}{1000}(3000) + 2$$
$$= -1$$
$$\not> 0$$

*A* is increasing on the interval $(1000, 2000)$.
*A* is decreasing on the interval $(2000, 3100)$.

**c)**

($x \in [0, 3100]$, Xscl = 500; $y \in [0, 1000]$, Yscl = 150)

**d)** An airplane has a peak efficiency engine speed (here about 2000 rpm). If that engine speed is exceeded, the engine will start to use more fuel per mile and hence the range will be shorter since there is a limited amount of fuel in the tank.

"There is no question indicated in the textbook for part d)."

**Chapter 3 Section 1 Question 15 Page 158**

The graph of $f$ is a parabola opening up since the coefficient of $x^2$ is positive (+3).
Such a curve will be decreasing to the left of the vertex and increasing to the right of it.
The interval $a < x < \infty$ must include some part of the right half of the parabola and so must include an increasing section. The function cannot be strictly decreasing on this interval.

**Chapter 3 Section 1 Question 16 Page 158**

$$f(x) = x^3 + bx^2 + 12x - 3$$
$$f'(x) = 3x^2 + 2bx + 12$$

$f$ is increasing when $f'(x) = 3x^2 + 2bx + 12 > 0$.

$$3x^2 + 2bx + 12 > 0$$

$$x^2 + \frac{2}{3}bx + 4 > 0 \qquad \text{Divide both sides by 4.}$$

$$\left(x^2 + \frac{2}{3}bx + \left(\frac{b}{3}\right)^2\right) - \left(\frac{b}{3}\right)^2 + 4 > 0 \qquad \text{Complete the square.}$$

$$\left(x + \frac{b}{3}\right)^2 - \frac{b^2}{9} + 4 > 0$$

$$\left(x + \frac{b}{3}\right)^2 > \frac{b^2}{9} - 4 \qquad \text{Add/subtract same amount on each side.}$$

The last inequality will be true for **all** values of $x$ if and only if the R.S. is less than zero.

$$\frac{b^2}{9} - 4 < 0$$

$$\frac{b^2}{9} < 4$$

$$b^2 < 36$$

Therefore, $-6 < b < 6$ is the necessary condition for the function *f* to be increasing for all values of *x*.

**Chapter 3 Section 1 Question 17 Page 158**

D is the correct answer.

$$y = x^{2n+1} + x^{2n-1} + \cdots + x^3 + x$$

$$\frac{dy}{dx} = (2n+1)x^{2n} + (2n-1)x^{2n-2} + \cdots + 3x^2 + 1$$

This expression is always positive since the exponents on *x* are always even and the coefficients of each term are positive (i.e., the odd numbers from 1 to 2*n* +1).
Since the derivative is always positive, the function is always increasing.

**Chapter 3 Section 1 Question 18 Page 158**

C is the correct answer.

If *f* is even, then $f(-x) = f(x)$.
Differentiate both sides of this equation using the chain rule.
$-f'(-x) = f'(x)$
Hence $f'$ is odd.

If *f* is odd, then $f(-x) = -f(x)$.
Differentiate both sides of this equation using the chain rule.
$-f'(-x) = -f'(x)$
Hence $f'$ is even.

**Chapter 3 Section 2 Maxima and Minima**

**Chapter 3 Section 2 Question 1 Page 163**

**a)** The absolute maximum value is 10 and the absolute minimum value is –3. (Assume the function is restricted to the domain shown.)

**b)** The absolute maximum value is 0.5 and the absolute minimum value is 0. (Assume the function is restricted to the domain shown.)

**Chapter 3 Section 2 Question 2 Page 163**

**a)** Since the function is linear, there are no critical numbers.
Examine the endpoints.
At $x = -10$,
$y = -(-10) + 7$
$= 17$

At $x = 10$,
$y = -(10) + 7$
$= -3$

The absolute maximum value is 17, occurring at the left endpoint.
The absolute minimum value is –3, occurring at the right endpoint.
There are no local extreme values.

**b)** Find the critical numbers.

$$f(x) = 3x^2 - 12x + 7$$
$$f'(x) = 6x - 12$$
$$6x - 12 = 0$$
$$x = 2$$

Examine the local extremum at $x = 2$ and the endpoints.

$$f(0) = 3(0)^2 - 12(0) + 7$$
$$= 7$$
$$f(2) = 3(2)^2 - 12(2) + 7$$
$$= -5$$
$$f(4) = 3(4)^2 - 12(4) + 7$$
$$= 7$$

The absolute maximum value is 7 and occurs twice, at both endpoints.
The absolute minimum value is –5 and occurs when $x = 2$.

**c)** Find the critical numbers.

$$f(x) = 2x^3 - 3x^2 - 12x + 2$$
$$f'(x) = 6x^2 - 6x - 12$$
$$6x^2 - 6x - 12 = 0$$
$$x^2 - x - 2 = 0$$
$$(x-2)(x+1) = 0$$

$$x = 2, x = -1$$

Examine the local extrema at $x = 2$ and $x = -1$ and the endpoints.

$$f(-3) = 2(-3)^3 - 3(-3)^2 - 12(-3) + 2$$
$$= -43$$
$$f(-1) = 2(-1)^3 - 3(-1)^2 - 12(-1) + 2$$
$$= 9$$
$$f(2) = 2(2)^3 - 3(2)^2 - 12(2) + 2$$
$$= -18$$
$$f(3) = 2(3)^3 - 3(3)^2 - 12(3) + 2$$
$$= -7$$

The absolute maximum value is 9.
The absolute minimum value is –43.
There is a local minimum value of –18 when $x = 2$.

**d)** Find the critical numbers.

$$f(x) = x^3 + x$$
$$f'(x) = 3x^2 + 1$$

$$3x^2 + 1 = 0$$
$$x^2 = -\frac{1}{3}$$

There are no solutions to this equation and so there are no critical points.

Examine the endpoints.

$$f(0) = (0)^3 + (0)$$
$$= 0$$
$$f(10) = (10)^3 + (10)$$
$$= 1010$$

The absolute maximum value is 1010.
The absolute minimum value is 0.
There are no local extrema.

**e)** Find the critical numbers.

$$f(x) = (x-3)^2 - 9$$
$$f'(x) = 2(x-3)(1)$$
$$= 2x - 6$$

$$2x - 6 = 0$$
$$x = 3$$

This critical point is not within the given interval.

Examine the endpoints.

$$f(-8) = ((-8)-3)^2 - 9$$
$$= 112$$
$$f(-3) = ((-3)-3)^2 - 9$$
$$= 27$$

The absolute maximum value is 112.
The absolute minimum value is 27.
There are no local extrema.

**Chapter 3 Section 2 Question 3 Page 163**

**a)**

$$f(x) = -x^2 + 6x + 2$$
$$f'(x) = -2x + 6$$

$$-2x + 6 = 0$$
$$x = 3$$

The critical number is 3.

**b)**

$$f(x) = x^3 - 2x^2 + 3x$$
$$f'(x) = 3x^2 - 4x + 3$$

$$3x^2 - 4x + 3 = 0$$
$$x = \frac{4 \pm \sqrt{(-4)^2 - 4(3)(3)}}{2(3)}$$
$$= \frac{4 \pm \sqrt{-20}}{6}$$

There are no critical numbers for this function.

**c)** $f(x) = x^4 - 3x^3 + 5$

$f'(x) = 4x^3 - 9x^2$

$4x^3 - 9x^2 = 0$

$x^2(4x - 9) = 0$

$x = 0, x = \frac{9}{4}$

The critical numbers are 0 and 2.25.

**d)** $g(x) = 2x^3 - 3x^2 - 12x + 5$

$g'(x) = 6x^2 - 6x - 12$

$6x^2 - 6x - 12 = 0$

$6(x^2 - x - 2) = 0$

$6(x - 2)(x + 1) = 0$

$x = -1,\ x = 2$

The critical numbers are –1 and 2.

**e)** $f(x) = x - \sqrt{x}$

$= x - x^{\frac{1}{2}}$

$f'(x) = 1 - \left(\frac{1}{2}\right) x^{-\frac{1}{2}}$

$= 1 - \frac{1}{2\sqrt{x}}$

$1 - \frac{1}{2\sqrt{x}} = 0$

$2\sqrt{x} - 1 = 0$

$\sqrt{x} = \frac{1}{2}$

$x = \frac{1}{4}$

The critical number is 0.25.

Note that $x = 0$ is not a critical number since that number is not in the domain of the function.

**Chapter 3 Section 2 Question 4 Page 164**

**a)**
$$\begin{aligned} y &= f(x) \\ &= 4x - x^2 \\ f'(x) &= 4 - 2x \end{aligned}$$

$$\begin{aligned} 4 - 2x &= 0 \\ x &= 2 \end{aligned}$$

Use a table to show increasing and decreasing intervals for the function.

| | $x < 2$ | $x = 2$ | $x > 2$ |
|---|---|---|---|
| **Test Value** | 0 | 2 | 4 |
| $f'(x)$ | $4 - 2(0) = 4$ Positive | 0 | $4 - 2(4) = -4$ Negative |
| $f(x)$ | Increasing | (2, 4) | Decreasing |

The critical point (2, 4) is a local maximum.

**b)**
$$\begin{aligned} f(x) &= (x-1)^4 \\ f'(x) &= 4(x-1)^3 \end{aligned}$$

$$\begin{aligned} 4(x-1)^3 &= 0 \\ x &= 1 \end{aligned}$$

Use a table to show increasing and decreasing intervals for the function.

| | $x < 1$ | $x = 1$ | $x > 1$ |
|---|---|---|---|
| **Test Value** | 0 | 1 | 4 |
| $f'(x)$ | $4((0)-1)^3 = -4$ Negative | 0 | $4((4)-1)^3 = 108$ Positive |
| $f(x)$ | Decreasing | (1, 0) | Increasing |

The critical point (1, 0) is a local minimum.

**c)**
$$\begin{aligned} g(x) &= 2x^3 - 24x + 5 \\ g'(x) &= 6x^2 - 24 \end{aligned}$$

$$\begin{aligned} 6x^2 - 24 &= 0 \\ x^2 &= 4 \\ x &= \pm 2 \end{aligned}$$

Use a table to show increasing and decreasing intervals for the function.

| | $x < -2$ | $x = -2$ | $-2 < x < 2$ | $x = 2$ | $x > 2$ |
|---|---|---|---|---|---|
| **Test Value** | –3 | –2 | 0 | 2 | 3 |
| $g'(x)$ | $g'(-3) = 30$ Positive | 0 | $g'(0) = -24$ Negative | 0 | $g'(3) = 30$ Positive |
| $g(x)$ | Increasing | (–2, 37) | Decreasing | (2, –27) | Increasing |

The critical point (–2, 37) is a local maximum and the critical point (2, –27) is a local minimum.

**d)** $h(x) = x^5 + x^3$

$h'(x) = 5x^4 + 3x^2$

$5x^4 + 3x^2 = 0$

$x^2(5x^2 + 3) = 0$

$x = 0$ There are no roots associated with the second factor.

Use a table to show increasing and decreasing intervals for the function.

| | $x < 0$ | $x = 0$ | $x > 0$ |
|---|---|---|---|
| **Test Value** | –1 | 0 | 1 |
| $h'(x)$ | $h'(-1) = 8$ Positive | 0 | $h'(1) = 8$ Positive |
| $h(x)$ | Increasing | (0, 0) | Increasing |

The critical point (0, 0) is neither a local minimum nor a local maximum.

**Chapter 3 Section 2 Question 5 Page 164**

When you stop to rest, your elevation is not changing for a period of time. The rate of change of elevation, $f'(t)$, is zero.

You would be at a local maximum if the trial from the rest point went down in all directions.
You would be at a local minimum if the trial from the rest point went up in all directions.
You would be at neither a maximum nor a minimum if the trail went up in one direction but down in the other, or if you were riding on level ground.

**Chapter 3 Section 2 Question 6 Page 164**

**a)** Find the critical numbers.

$f(x) = 2x^3 - 3x^2 - 12x + 5$

$f'(x) = 6x^2 - 6x - 12$

$6x^2 - 6x - 12 = 0$

$x^2 - x - 2 = 0$

$(x - 2)(x + 1) = 0$

$x = -1, x = 2$

The critical numbers are –1 and 2.

**b)** To find local extrema, use a table to show increasing and decreasing intervals for the function.

| | $x < -1$ | $x = -1$ | $-1 < x < 2$ | $x = 2$ | $x > 2$ |
|---|---|---|---|---|---|
| **Test Value** | –2 | –1 | 0 | 2 | 3 |
| $f'(x)$ | $f'(-2) = 24$ Positive | 0 | $f'(0) = -12$ Negative | 0 | $f'(3) = 24$ Positive |
| $f(x)$ | Increasing | (–1, 12) | Decreasing | (2, –15) | Increasing |

The critical point (–1, 12) is a local maximum and the critical point (2, –15) is local minimum.

**c)** Check the function values at the endpoints of the interval.

$f(-2) = 1$

$f(4) = 37$

The absolute minimum is (2, –15) and the absolute maximum is (4, 37) for the interval $[-2, 4]$.

**Chapter 3 Section 2 Question 7 Page 164**

**a)** Find the critical values.

$f(x) = 7 + 6x - x^2$

$f'(x) = 6 - 2x$

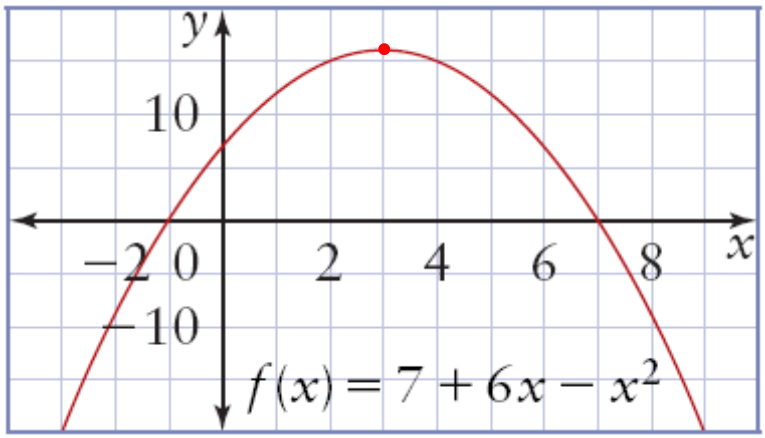
$6 - 2x = 0$

$x = 3$

To decide the nature of this critical value, evaluate $f'(x)$. This information can be summarized in a table.

| | $x < 3$ | $x = 3$ | $x > 3$ |
|---|---|---|---|
| Test value | 0 | 3 | 4 |
| $f'(x)$ | $f'(0) = 6$ Positive | 0 | $f'(4) = -2$ Negative |
| $f(x)$ | Increasing | (3, 16) | Decreasing |

Sketch a curve with an absolute maximum at (3, 16).




**b)** Find the critical values.

$g(x) = x^4 - 8x^2 - 10$

$g'(x) = 4x^3 - 16x$

$4x^3 - 16x = 0$

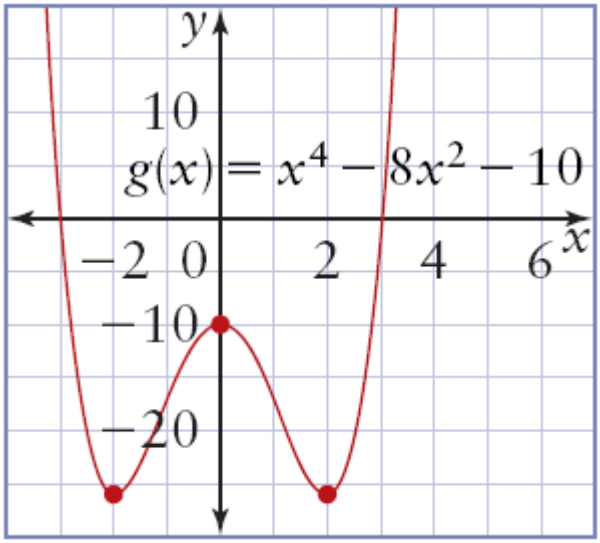
$4x\left(x^2 - 4\right) = 0$

$x = 0,\ x = -2,\ x = 2$

To decide the nature of these critical values, evaluate $g'(x)$. This information can be summarized in a table.

| | $x < -2$ | $x = -2$ | $-2 < x < 0$ | $x = 0$ | $0 < x < 2$ | $x = 2$ | $x > 2$ |
|---|---|---|---|---|---|---|---|
| **Test Value** | –3 | –2 | –1 | 0 | 1 | 2 | 3 |
| $g'(x)$ | $g'(-3) = -60$ Negative | 0 | $g'(-1) = 12$ Positive | 0 | $g'(1) = -12$ Negative | 0 | $g'(3) = 60$ Positive |
| $g(x)$ | Decreasing | (–2, –26) | Increasing | (0, –10) | Decreasing | (2, –26) | Increasing |

Sketch a curve with an absolute minima at (–2, –26) and (2, –26) and a local maximum at (0, –10).




**c)** Find the critical values.

$$\begin{aligned} y &= f(x) \\ &= x(x+2)^2 \\ &= x^3 + 4x^2 + 4x \end{aligned}$$

$$f'(x) = 3x^2 + 8x + 4$$

$$3x^2 + 8x + 4 = 0$$

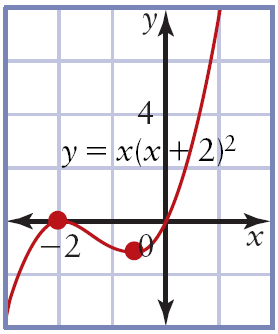
$$(3x+2)(x+2) = 0$$

$$x = -\frac{2}{3},\ x = -2$$

To decide the nature of these critical values, evaluate $f'(x)$ . This information can be summarized in a table.

| | $x < -2$ | $x = -2$ | $-2 < x < -\frac{2}{3}$ | $x = -\frac{2}{3}$ | $x > -\frac{2}{3}$ |
|---|---|---|---|---|---|
| **Test Value** | –3 | –2 | –1 | $-\frac{2}{3}$ | 0 |
| $f'(x)$ | $f'(-3) = 7$ Positive | 0 | $f'(-1) = -1$ Negative | 0 | $f'(0) = 4$ Positive |
| $f(x)$ | Increasing | (–2, 0) | Decreasing | $\left(-\frac{2}{3}, -\frac{32}{27}\right)$ | Increasing |

Sketch a curve with a local maximum at (–2, 0) and a local minimum at $\left(-\frac{2}{3}, -\frac{32}{27}\right)$.




**d)** Find the critical values.

$h(x) = 27x - x^3$

$h'(x) = 27 - 3x^2$

$27 - 3x^2 = 0$

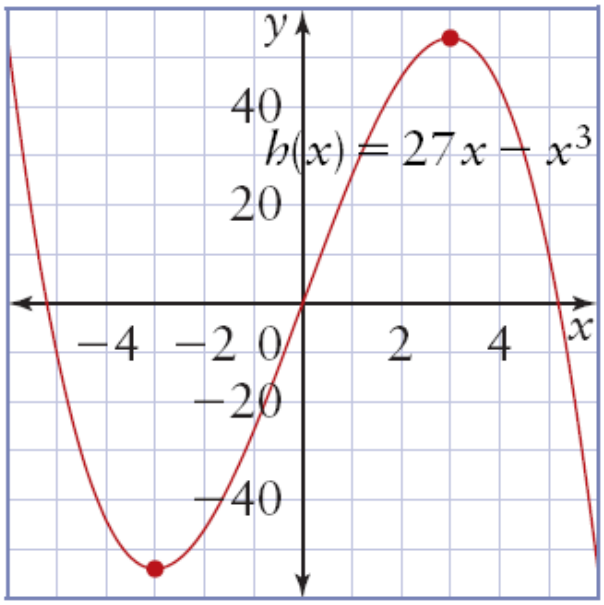
$x^2 = 9$

$x = -3,\ x = 3$

To decide the nature of these critical values, evaluate $h'(x)$ . This information can be summarized in a table.

| | $x < -3$ | $x = -3$ | $-3 < x < 3$ | $x = 3$ | $x > 3$ |
|---|---|---|---|---|---|
| **Test Value** | –4 | –3 | 0 | 3 | 4 |
| $h'(x)$ | $h'(-4) = -21$ Negative | 0 | $h'(0) = 27$ Positive | 0 | $h'(4) = -21$ Negative |
| $h(x)$ | Decreasing | (–3, –54) | Increasing | (3, 54) | Decreasing |

Sketch a curve with a local minimum at (–3, –54) and a local maximum at (3, 54).




**Chapter 3 Section 2 Question 8 Page 164**

No, the statement is generally false. There could be any number of local minima and maxima between $x = a$ and $x = b$. The function can have many sections of increase and decrease in the specified interval provided that the absolute maximum is at $x = a$ and the absolute minimum is at $x = b$.

**Chapter 3 Section 2 Question 9 Page 164**

**a)** The vertex of the parabola $f(x)$ is (3, 0) and the direction of opening is up.

**b)** $f'(x) = 2(x-3)(1)$

$$2(x-3) = 0$$
$$x = 3$$

Check the value of $f'(x)$ in the specified interval.
$f'(4) = 2$

Since this value is positive, $f(x)$ is increasing for $3 < x < 6$.
Therefore $x = 3$ is a global minimum value and $x = 6$ is a global maximum value.

**c)** The function is a parabola opening up, having vertex (3, 0).
The interval $3 < x < 6$ only involves the right half of the graph which is increasing only.
The global minimum value must occur at the start of the interval while the global maximum must occur at the end of the interval.

**Chapter 3 Section 2 Question 10 Page 164**

**a)** $x^3 - 2x^2 = 0$

$x^2(x-2) = 0$

**b)** The critical points create 3 intervals. Test the derivative in each interval.

| | $x < 0$ | $0 < x < 2$ | $x > 2$ |
|---|---|---|---|
| **Test Value** | $-1$ | $1$ | $3$ |
| $f'(x)$ | $f'(1) = -3$ Negative | $f'(1) = -1$ Negative | $f'(3) = 9$ Positive |
| $f(x)$ | Decreasing | Decreasing | Increasing |

The function is decreasing for $x < 0$ and $0 < x < 2$ and is increasing for $x > 2$.

**c)** Polynomial functions usually have a turning point at each critical value. That is because the function changes from increasing to/from decreasing at these points. The 'double root' $x = 0$ leads to the function not changing from increase/decrease at that point and therefore there is no turning point there. The only turning point occurs when $x = 2$.

**Chapter 3 Section 2 Question 11 Page 164**

**a)** Find the critical numbers.

$f(x) = x^3 - 6x^2 + 11x$

$f'(x) = 3x^2 - 12x + 11$

$3x^2 - 12x + 11 = 0$

$$x = \frac{12 \pm \sqrt{(-12)^2 - 4(3)(11)}}{2(3)}$$

$$= \frac{12 \pm \sqrt{12}}{6}$$

$$= \frac{6 \pm \sqrt{3}}{3}$$

$x \approx 2.6,\ x \approx 1.4$

The critical numbers are $-\frac{6+\sqrt{3}}{3}$ and $\frac{6-\sqrt{3}}{3}$.

**b)** Test the function at the critical points and at the endpoints of the interval.

$f(0) = 0$

$f(1.4) \approx 6.4$

$f(2.6) \approx 5.6$

$f(4) = 12$

The absolute minimum value is 0; the absolute maximum value is 4.

**Chapter 3 Section 2** **Question 12 Page 164**

**a)** $A(x)=\left(\frac{4+\pi}{4\pi}\right)x^2-10x+100$

$A'(x)=\left(\frac{4+\pi}{2\pi}\right)x-10$

$\left(\frac{4+\pi}{2\pi}\right)x-10=0$

$x=\frac{20\pi}{4+\pi}$

There is only one critical number: $\frac{20\pi}{4+\pi}$.

**b)** $\frac{20\pi}{4+\pi}$ B 8.8

Use $x$ = 0 and $x$ = 10 for test values in the intervals.

| | $x<\frac{20\pi}{4+\pi}$ | $x=\frac{20\pi}{4+\pi}$ | $x>\frac{20\pi}{4+\pi}$ |
|---|---|---|---|
| $A'(x)$ | Negative | 0 | Positive |

**c)** The critical point is a local minimum because the function is decreasing on the interval before it and increasing in the interval after it.

**d)** Since the critical point is local minimum, check the endpoints for a maximum.

$A(5)$ B 64.2

$A(15)$ B 77.9

The maximum area for the interval $5\le x\le 15$ is about 77.9 m$^2$.

**Chapter 3 Section 2 Question 13 Page 164**

**a)** Find the critical points.

$$f(x) = -x^3 - 2x^2 + x + 15$$
$$f'(x) = -3x^2 - 4x + 1$$

$$-3x^2 - 4x + 1 = 0$$
$$3x^2 + 4x - 1 = 0$$

$$x = \frac{-4 \pm \sqrt{4^2 - 4(3)(-1)}}{6}$$
$$x = \frac{-4 \pm \sqrt{28}}{6}$$

$$x = 0.22,\ x = -1.55$$

To decide the nature of these critical values, evaluate $f'(x)$. This information can be summarized in a table.

" In the below table the $\sqrt{26}$ part should actually be $\sqrt{28}$. Please adjust all the tabular values accordingly."

| | $x < \frac{-4-\sqrt{28}}{6}$ | $x = \frac{-4-\sqrt{28}}{6}$ | $\frac{-4-\sqrt{28}}{6} < x < \frac{-4+\sqrt{28}}{6}$ | $x = \frac{-4+\sqrt{28}}{6}$ | $x > \frac{-4+\sqrt{28}}{6}$ |
|---|---|---|---|---|---|
| **Test Value** | $-2$ | $\frac{-4-\sqrt{28}}{6}$ | 0 | $\frac{-4+\sqrt{28}}{6}$ | 1 |
| $h'(x)$ | $f'(-2) = -3$ Negative | 0 | $f'(0) = 1$ Positive | 0 | $f'(1) = -6$ Negative |
| $h(x)$ | Decreasing | (–1.55, 12.37) | Increasing | (0.22, 15.11) | Decreasing |

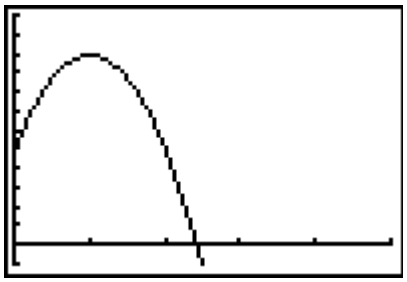
The coaster starts down a hill from $x = -2$, reaching a local minimum at the bottom of a hill at (–1.55, 12.37). It then increases height until it reaches a local maximum at the top of a hill at (0.22, 15.11). It then continues downward until $x = 2$.

**b)** Check the values at the endpoints.

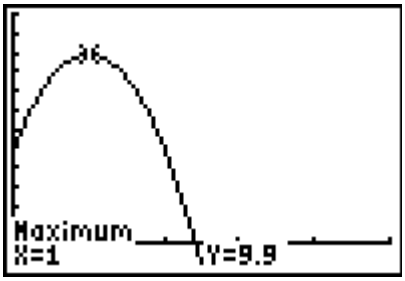
$$f(-2) = 13$$
$$f(2) = 1$$

The highest point of the ride is at the critical point (0.22, 15.11), not at either endpoint.

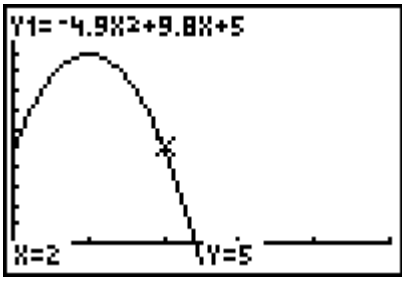
**Chapter 3 Section 2 Question 14 Page 164**

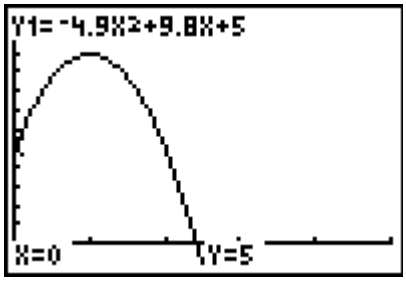


**a)** Use the **maximum** operation on the graphing calculator.




The minimum value occurs at an endpoint. Use the **value** operation to determine this value.




The minimum height is 5 m and the maximum height is 9.9 m.

**b)** One method is shown in part a) of the question. Some others include:

- Use the **trace** operation to trace the curve at its highest and lowest points.
- Use the [TABLE] feature to examine a table of values of the function.
- Graph $h'(t)$ and find the **zeroes**.
- Graph $h(t)$; then [ZOOM] in to find the coordinates of the extrema.

**c)** Answers will vary. Some possibilities include:

- Fill in random values of *t* until you converge on the maximum or minimum.
- Set $h(t) = 5$ and solve for *t*. The *t*-value that is half way between your two values will be the *t*-value of the maximum.
- Complete the square to determine the vertex. The curve is a parabola that will have a maximum value at its vertex.

**Chapter 3 Section 2 Question 15 Page 165**

**a)** Find the critical points for this function.

$$d(v) = 4.8v^3 - 28.8v^2 + 52.8v$$
$$d'(v) = 14.4v^2 - 57.6v + 52.8$$

$$14.4v^2 - 57.6v + 52.8 = 0$$
$$3v^2 - 12v + 11 = 0$$

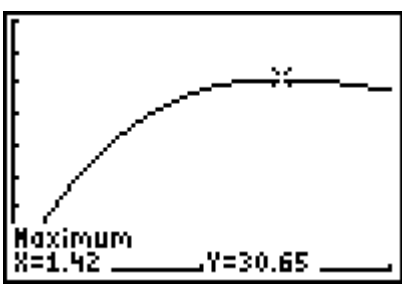
$$v = \frac{12 \pm \sqrt{12}}{6}$$
$$v = \frac{6 \pm \sqrt{3}}{3}$$
$$v \approx 2.58,\ v = 1.42$$

Only one of these critical points is in the interval. Check for the maximum distance by substituting that value and also for the endpoints in the function.

$$d(0) = 0$$
$$d\left(\frac{6-\sqrt{3}}{3}\right) \approx 30.65$$
$$d(2) = 28.8$$

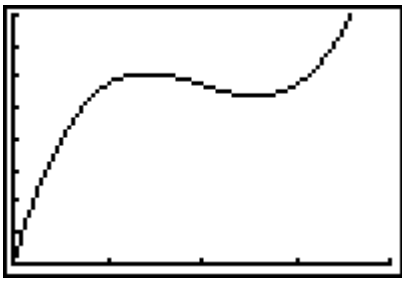
A speed of about 1.42 m/s will result in a maximum distance that the diver can swim.

**b)** The graphing calculator verifies our algebraic result.



(Yscl: 5)

**c)** If you extend the domain of the graph, the function increases without bound after $x = 2$. This is not reasonable since a swimmer has a limit to how fast the person can swim. A speed of 2m/s is already quite fast and sustainable only over very short distances.


(Yscl:5)

**Chapter 3 Section 2 Question 16 Page 165**

Solutions for Achievement Checks are shown in the Teacher's Resource.

**Chapter 3 Section 2 Question 17 Page 165**

$f(x) = ax^4 + bx^2 + cx + d$
$f'(x) = 4ax^3 + 2bx + c$

Since there are critical value at $x = 0$ and $x = 1$, the cubic derivative function must be of the form $f'(x) = kx(x-1)(x-p)$ where $p$ is a third critical number and $k$ must be $4a$.

$4ax^3 + 2bx + c = 4ax(x-1)(x-p)$
$4ax^3 + 2bx + c = 4ax^3 + 4a(p-1)x^2 - 4apx$

Comparing coefficients of like terms,
$0 = 4a(p-1)$ $x^2$ terms
$p = 1$

$2b = -4ap$
$b = -2a$ $x$ terms $(p = 1)$
$c = 0$ constant terms

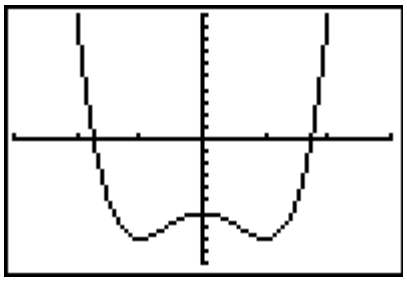
Therefore the function is now of the form $f(x) = ax^4 - 2ax^2 + d$.

Since (0, –6) is on the curve,
$a(0)^4 - 2a(0)^2 + d = -6$
$d = -6$

Since (1, –8) is on the curve,
$a(1)^4 - 2a(1)^2 - 6 = -8$
$a = 2$

Therefore $f(x) = 2x^4 - 4x^2 - 6$; that is, $a = 2$, $b = -4$, $c = 0$, and $d = -6$.

A graph of $f(x) = 2x^4 - 4x^2 - 6$ confirms the solution.



**Chapter 3 Section 2 Question 18 Page 165**

**a)** Find the critical numbers.

$f(x) = ax^3 + bx^2 + cx + d$

$f'(x) = 3ax^2 + 2bx + c$

$3ax^2 + 2bx + c = 0$

$$x = \frac{-2b \pm \sqrt{(2b)^2 - 4(3a)(c)}}{6a}$$

There will be no critical numbers and hence no extrema if the discriminant is negative.

$4b^2 - 12ac < 0$

$b^2 - 3ac < 0$ or $b^2 < 3ac$

There are no extrema when $b^2 < 3ac$ or $b < \sqrt{3ac}$.

**b)** There will be exactly two critical numbers and hence two extrema if the discriminant is strictly positive.

$4b^2 - 12ac > 0$

$b^2 - 3ac > 0$ or $b^2 > 3ac$

There are exactly two extrema when $b^2 > 3ac$ or $b > \sqrt{3ac}$.

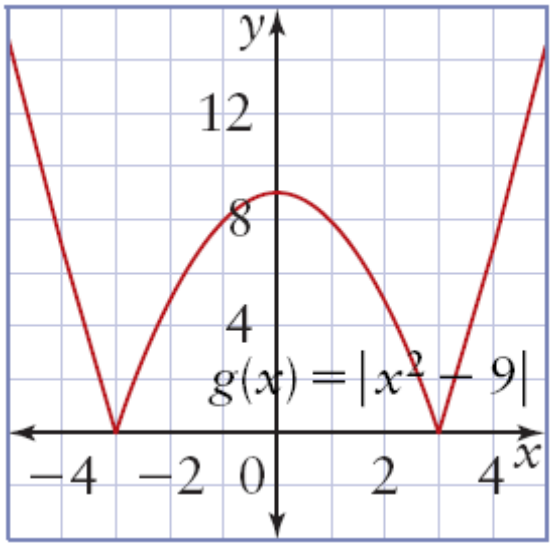
**Chapter 3 Section 2 Question 19 Page 165**

To find possible extrema, we examine the derivative of the function.
The derivative of a cubic function is quadratic.
This derivative will have either
**i)** no (real) roots,
**ii)** two identical (real) roots, or
**iii)** two different (real) roots.

In the first case, there will be no extrema since there are no roots.
In the third case, there will be exactly two extrema.
In the second case, there will also be no extrema. At each critical point, a polynomial function changes from increasing to decreasing or decreasing to increasing. At a double root, a polynomial function changes twice and so does not change. Such a critical number does not lead to an extremum.

**Chapter 3 Section 2 Question 20 Page 165**

**a)** Graph $y = x^2 - 9$. Any points below the $x$-axis will be reflected in the $x$-axis to produce the graph of $g(x) = |x^2 - 9|$.




**b)** Critical points occur when the derivative is zero or undefined.
Find the zeros of $y = x^2 - 9$.

$y' = 2x$

$2x = 0$

$x = 0$

One critical point occurs when $x = 0$.

Examine values of $y'$ before and after 0. The point (0, –9) is an (absolute) minimum point for $y = x^2 - 9$. However, reflection in the $x$-axis reverses this to become a (local) maximum point (0, 9) for $g(x) = |x^2 - 9|$.
The derivative of $g(x) = |x^2 - 9|$ is undefined where $y = x^2 - 9$ has an $x$-intercept, i.e., at $x = -3$ and $x = 3$. Since $g(x) = |x^2 - 9|$ is always above the $x$-axis, (–3, 0) and (3, 0) must be (absolute) minimum points.

**c)** Consider $g(x)$ as a piecewise function.

$$g(x) = \begin{cases} x^2 - 9 & \text{if} \quad x < -3 \\ -x^2 + 9 & \text{if} \quad -3 \le x \le 3 \\ x^2 - 9 & \text{if} \quad x > 3 \end{cases}$$

Since each of the pieces is a polynomial, the derivative is easily computed. At the boundaries for the intervals, the derivative will be undefined since the left and right hand derivatives are not equal.

$$g'(x)=\begin{cases}2x & \text{if} \quad x<-3\\ \text{undefined} & \text{if} \quad x=-3\\ -2x & \text{if} \quad -3\le x\le 3\\ \text{undefined} & \text{if} \quad x=3\\ 2x & \text{if} \quad x>3\end{cases}$$

**Chapter 3 Section 2 Question 21 Page 165**

B is the correct answer.

$y=x^n-nx$

$\frac{dy}{dx}=nx^{n-1}-n$

For all positive integers, $n\ge 2$, $\frac{dy}{dx}=0$ when $x=1$.

Note that $\frac{dy}{dx}=0$ when $x=-1$ only for odd *n*.
Hence, depending on the parity of *n*, the function does not necessarily have a critical point at $x=-1$.

For $x=1^-$, $\frac{dy}{dx}<0$; for $x=1^+$, $\frac{dy}{dx}>0$.
Therefore, there is a local minimum at $x=1$.

**Chapter 3 Section 2 Question 22 Page 165**

D is the correct answer.

The function $f(x)$ has a local minimum at $x=a$ because $f'(x)$ changes from negative to positive when passing through $x=a$.

**Chapter 3 Section 2 Question 23 Page 165**

E is the correct answer.

The function $f(x)$ has a horizontal tangent at $x=b$ because $f'(b)=0$.
Note that $f(x)$ is increasing on either side of $x=b$.

**Chapter 3 Section 3 Concavity and the Second Derivative Test**

**Chapter 3 Section 3 Question 1 Page 173**

**a)** Graph is concave up when $x > -2$.
Graph is concave down when $x < -2$.

**b)** Graph is concave up when $x < -2$ and $x > 0$.
Graph is concave down when $-2 < x < 0$.

**Chapter 3 Section 3 Question 2 Page 173**

**a)** The function $f(x)$ is concave up for $x \in \mathbb{R}$. There are no points of inflection.

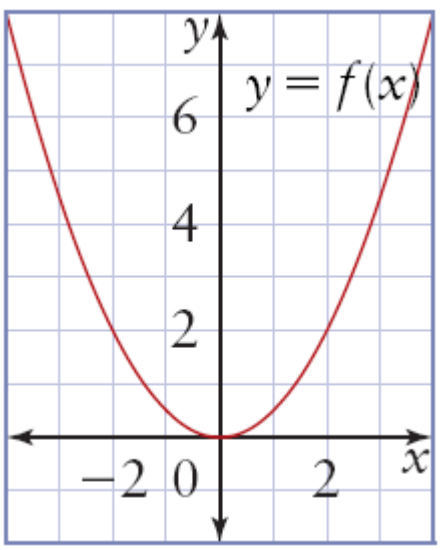
**b)** The function $f(x)$ is concave up for $x > 2$ and concave down for $x < 2$. There is a point of inflection when $x = 2$.

**c)** The function $f(x)$ is concave up for $-1 < x < 2$ and concave down for $x < -1$ and $x > 2$. There are points of inflection when $x = -1$ and $x = 2$.

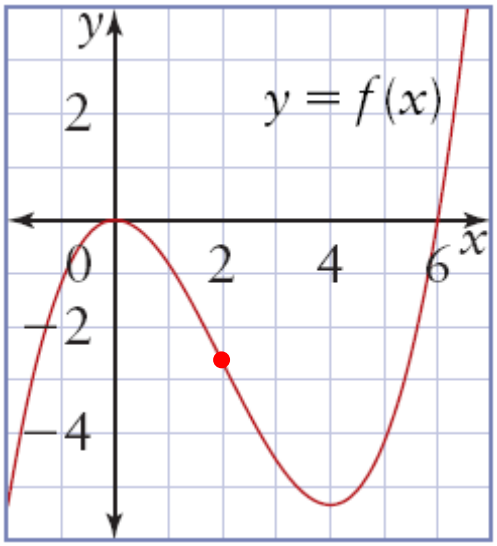
**d)** The function $f(x)$ is concave up for $x > 2$ and concave down for $x < -1$ and $-1 < x < 2$. There is a point of inflection when $x = 2$.

**Chapter 3 Section 3 Question 3 Page 174**

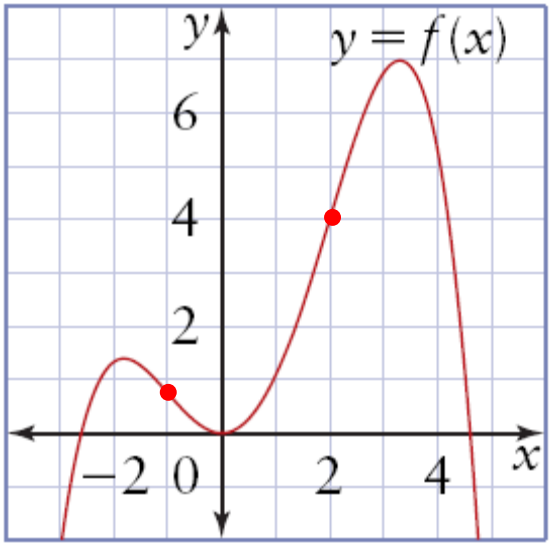
**a)**



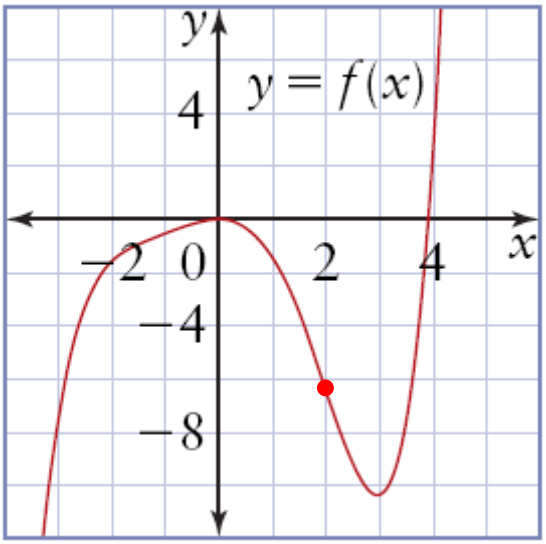
**b)**




**c)**



**d)**




**Chapter 3 Section 3** **Question 4 Page 174**

**a)** $y = 6x^2 - 7x + 5$

$y' = 12x - 7$

$y'' = 12$

**b)** $f(x) = x^3 + x$

$f'(x) = 3x^2 + 1$

$f''(x) = 6x$

**c)** $g(x) = -2x^3 + 12x^2 - 9$

$g'(x) = -6x^2 + 24x$

$g''(x) = -12x + 24$

**d)** $y = x^6 - 5x^4$

$y' = 6x^5 - 20x^3$

$y'' = 30x^4 - 60x^2$

**Chapter 3 Section 3** **Question 5 Page 174**

**a)** $y'' = 12$

This function is always concave up (i.e., for $x \in$ °  ). There are no points of inflection.

**b)** $f''(x) = 6x$

$6x = 0$

$x = 0$

This value divides the domain into two intervals. Test the value of $f''(x)$ in each interval and summarize in a table.

| | $x < 0$ | $x = 0$ | $x > 0$ |
|---|---|---|---|
| **Test Value** | $-1$ | 0 | 1 |
| $f''(x)$ | $f''(-1) = -6$ Negative | 0 | $f''(1) = 6$ Positive |
| $f(x)$ | Concave down | Point of inflection (0, 0) | Concave up |

**c)** $g''(x) = -12x + 24$

$-12x + 24 = 0$

$x = 2$

This value divides the domain into two intervals. Test the value of $f''(x)$ in each interval and summarize in a table.

| | $x < 2$ | $x = 2$ | $x > 2$ |
|---|---|---|---|
| **Test Value** | 0 | 2 | 3 |
| $g''(x)$ | $g''(0) = 24$ Positive | 0 | $g''(3) = -12$ Negative |
| $g(x)$ | Concave up | Point of inflection (2, 23) | Concave down |

**d)** $y'' = 30x^4 - 60x^2$

$30x^4 - 60x^2 = 0$

$30x^2(x^2 - 2) = 0$

$x = 0,\ x = -\sqrt{2},\ x = \sqrt{2}$

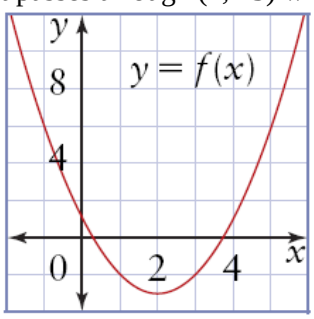
These values divide the domain into four intervals.

Test the value of $f''(x)$ in each interval and summarize in a table.

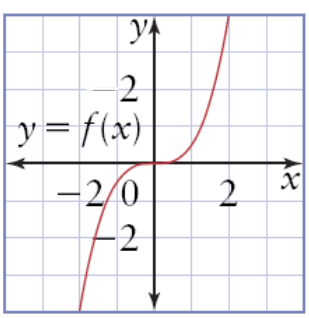
| | $x < -\sqrt{2}$ | $x = -\sqrt{2}$ | $-\sqrt{2} < x < 0$ | $x = 0$ | $0 < x < \sqrt{2}$ | $x = \sqrt{2}$ | $x > \sqrt{2}$ |
|---|---|---|---|---|---|---|---|
| **Test Value** | $-2$ | $-\sqrt{2}$ | $-1$ | $0$ | $1$ | $\sqrt{2}$ | $2$ |
| $y'' = f''(x)$ | $f''(-2)$ $= 240$ Positive | $0$ | $f''(-1)$ $= -30$ Negative | $0$ | $f''(1)$ $= -30$ Negative | $0$ | $f''(3)$ $= 240$ Positive |
| $y = f(x)$ | Concave up | Point of Inflection $\left(-\sqrt{2}, -12\right)$ | Concave down | No Point of Inflection | Concave down | Point of Inflection $\left(\sqrt{2}, -12\right)$ | Concave up |

**Chapter 3 Section 3** **Question 6 Page 174**

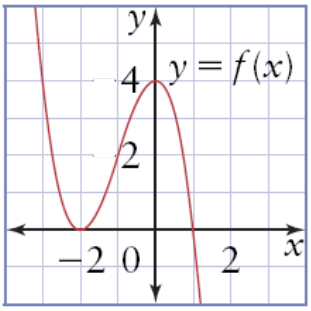
**a)** This function is always concave up.
It passes through (2, –3) where the slope is 0 (horizontal).



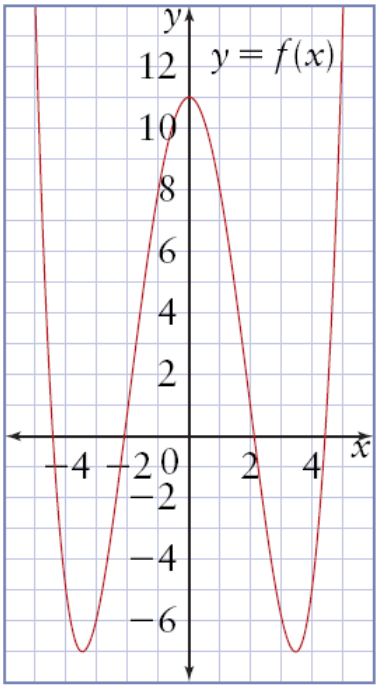

**b)** This function is concave down when $x < 0$ and concave up when $x > 0$.
It passes through the origin where it has a horizontal slope (point of inflection).



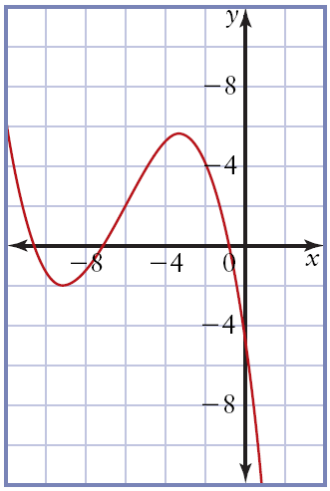
**c)** This function is concave up when $x < -1$ and concave down when $x > -1$.
It passes through (–1, 2) where it has a slope of 1.



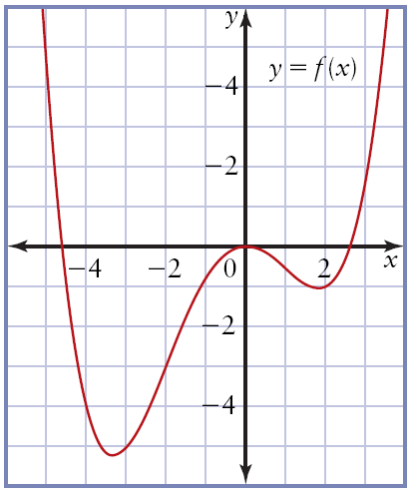

**d)** This function is concave down between –2 and 2 and concave up when $x < -2$ or $x > 2$.
It passes through (2, 1). Since the function is even, it is symmetric about the *y*-axis. For instance, it passes through (–2, 1).



**e)** This function is concave up when $x < -6$ and concave down when $x > -6$.
It passes through (–6, 2) where it has a slope of 2.




**f)** This function is concave down between –2 and 1 and concave up when $x < -2$ or $x > 1$.
It passes through (–2, –3) and the origin.




**Chapter 3 Section 3 Question 7 Page 174**

**a)** Begin as in section 2.2. Find the critical values.

$$y = f(x)$$
$$= x^2 + 10x - 11$$

$$f'(x) = 2x + 10$$

$$2x + 10 = 0$$
$$x = -5$$

Use a table to show increasing and decreasing intervals for the function.

| | $x < -5$ | $x = -5$ | $x > -5$ |
|---|---|---|---|
| **Test Value** | –6 | –5 | 0 |
| $f'(x)$ | $2(-6)+10=-2$ Negative | 0 | $2(0)+10=10$ Positive |
| $f(x)$ | Decreasing | (–5, –36) | Increasing |

The critical point (–5, –36) is an absolute minimum.

$f''(x)=2$, which is always positive. This means the curve is always concave up and any critical point must be a minimum.

**b)** Find the critical values.

$g(x)=3x^5-5x^3-5$

$g'(x)=15x^4-15x^2$

$15x^4-15x^2=0$

$15x^2(x^2-1)=0$

$x=0,\ x=-1,\ x=1$

Use a table to show increasing and decreasing intervals for the function.

| | $x < -1$ | $x = -1$ | $-1 < x < 0$ | $x = 0$ | $0 < x < 1$ | $x = 1$ | $x > 1$ |
|---|---|---|---|---|---|---|---|
| **Test Value** | –2 | –1 | –0.5 | 0 | 0.5 | 1 | 2 |
| $g'(x)$ | $g'(-2)=180$ Positive | 0 | $g'(-0.5)$ B –2.8 Negative | 0 | $g'(0.5)$ B –2.8 Negative | 0 | $g'(2)=180$ Positive |
| $g(x)$ | Increasing | (–1, –3) | Decreasing | (0, 0) | Decreasing | (1, –7) | Increasing |

The critical point (–1, –3) is a local maximum and the critical point (1, –7) is a local minimum.

Now use the second derivative test to verify the results.

$g''(x)=60x^3-30x$

Test the critical points.

$g''(-1)=-30$

$g''(0)=0$

$g''(1)=30$

The second derivative test indicates that there is a maximum at $x=-1$ and a minimum at $x=1$.
Since the sign of the second derivative changes at $x=0$, there is a point of inflection there.

**c)** Find the critical values.

$$f(x) = x^4 - 6x^2 + 10$$
$$f'(x) = 4x^3 - 12x$$

$$4x^3 - 12x = 0$$
$$4x(x^2 - 3) = 0$$

$$x = 0,\ x = -\sqrt{3},\ x = \sqrt{3}$$

Use a table to show increasing and decreasing intervals for the function.

| | $x < -\sqrt{3}$ | $x = -\sqrt{3}$ | $-\sqrt{3} < x < 0$ | $x = 0$ | $0 < x < \sqrt{3}$ | $x = \sqrt{3}$ | $x > \sqrt{3}$ |
|---|---|---|---|---|---|---|---|
| **Test Value** | $-2$ | $-\sqrt{3}$ | $-1$ | 0 | 1 | $\sqrt{3}$ | $2$ |
| $f'(x)$ | $f'(-2) = -8$ Negative | 0 | $f'(-1) = 8$ Positive | 0 | $f'(1) = -8$ Negative | 0 | $f'(2) = 8$ Positive |
| $f(x)$ | Decreasing | $(-\sqrt{3}, 1)$ | Increasing | (0, 10) | Decreasing | $(\sqrt{3}, 1)$ | Increasing |

The critical point $(-\sqrt{3}, 1)$ is a local minimum, (0, 10) is a local maximum and $(\sqrt{3}, 1)$ is a local minimum.

Now use the second derivative test to verify the results.

$$f''(x) = 12x^2 - 12$$

Test the critical points.

$$f''(-\sqrt{3}) = 24$$
$$f''(0) = 0$$
$$f''(\sqrt{3}) = 24$$

The second derivative test indicates that there are minimums at $x = -\sqrt{3}$ and $x = \sqrt{3}$ and a maximum at $x = 0$.

**d)** Find the critical values.

$$h(t) = -4.9t^2 + 39.2t + 2$$
$$h'(t) = -9.8t + 39.2$$

$$-9.8t + 39.2 = 0$$
$$t = 4$$

Use a table to show increasing and decreasing intervals for the function.

| | $t < 4$ | $t = 4$ | $t > 4$ |
|---|---|---|---|
| **Test Value** | 0 | 4 | 5 |
| $h'(t)$ | $h'(0) = 39.2$ Positive | 0 | $h'(5) = -9.8$ Negative |
| $h(t)$ | Increasing | (4, 80.4) | Decreasing |

The critical point (4, 80.4) is an absolute maximum.

Now use the second derivative test to verify the results.
$h''(t) = -9.8$ which is always negative.
The second derivative test indicates that there is a maximum at $t = 4$.

**Chapter 3 Section 3 Question 8 Page 174**

**a)** $h(x) = 0.01x^3 - 0.3x^2 + 60$

$h'(x) = 0.03x^2 - 0.6x$

$h''(x) = 0.06x - 0.6$

$0.06x - 0.6 = 0$

$x = 10$

Use a table to show concavity of the intervals for the function.

| | $0 < x < 10$ | $x = 10$ | $10 < x < 22$ |
|---|---|---|---|
| **Test Value** | 1 | 10 | 20 |
| $h''(x)$ | $h''(1) = -0.54$ Negative | 0 | $h''(20) = 0.6$ Positive |
| $h(x)$ | Concave down | Point of Inflection (10, 40) | Concave up |

**b)** The steepest slope occurs when $h'(x)$ has a maximum. This will occur when its derivative, $h''(x)$, equals zero. This occurs when $x = 10$, the inflection point.
The steepest point on the ski ramp is at (10, 40).

**Chapter 3 Section 3 Question 9 Page 174**

**a)** For a polynomial function, this is sometimes true. The exception is that if the minimum or maximum occurs at an endpoint of an interval, the derivative will not be defined for that point.
This statement may not be true for other types of functions, such as broken line functions like $f(x) = |x|$ where the derivative is undefined at some local extrema.

**b)** This is always true. The definition of inflection point requires that the curve be concave up and down on opposite sides of the point. This requires the second derivative to be zero at the point of inflection.

**Chapter 3 Section 3 Question 10 Page 174**

**a)** The curve is concave up for all values of $x$ since this is a parabola with the coefficient of $x^2$ positive.

**b)** The curve will have a local minimum since it is a parabola opening up. The minimum is located at the vertex. You can determine this point using calculus.

$$A(x) = \left(\frac{4+\pi}{4\pi}\right)x^2 - 10x + 100$$

$$A'(x) = \left(\frac{4+\pi}{2\pi}\right)x - 10$$

$$\left(\frac{4+\pi}{2\pi}\right)x - 10 = 0$$

$$x = \frac{20\pi}{4+\pi}$$

The (absolute) minimum point is $\left(\frac{20\pi}{4+\pi}, \frac{400}{4+\pi}\right)$.

**c)** The maximum area must occur at one of the endpoints of the interval since the curve is concave up throughout the interval.

$A(0) = 100$

$A(20) \approx 127.3$

The maximum occurs when $x = 20$.

**Chapter 3 Section 3 Question 11 Page 174**

The car is starting out at A and then accelerating over the interval AB. The driver takes his foot off the gas at B and then decelerates over the interval BC. At C the car is stopped. The driver then puts on the gas and accelerates over the interval CD. The driver then releases the gas at D and decelerates over the interval DE. At E the driver stops and turns the car around. Then he accelerates from E to F. At F the driver stops accelerating and decelerates over the interval FG. At G the car is back at the original location and the driver stops.

**Chapter 3 Section 3 Question 12 Page 175**

**a)** Find the critical numbers.

$$T(t) = -0.0003t^3 + 0.012t^2 - 0.112t + 36$$

$$T'(t) = -0.0009t^2 + 0.024t - 0.112$$

$$T''(t) = -0.0018t + 0.024$$

For critical numbers,

$$-0.0009t^2 + 0.024t - 0.112 = 0$$

$$9t^2 - 240t + 1120 = 0$$

$$t = \frac{240 \pm \sqrt{(-240)^2 - 4(9)(1120)}}{2(9)}$$

$$t = \frac{240 \pm \sqrt{17\ 280}}{18}$$

$$t = \frac{40 \pm 4\sqrt{30}}{3}$$

$$t \approx 20.6,\ t \approx 6.0$$

The critical numbers are 6.0 and 20.6 days.

**b)** The rate of change of temperature is a maximum when $T''(t) = 0$.

$$-0.0018t + 0.024 = 0$$

$$t = \frac{40}{3}$$

$$t \approx 13.3$$

The female is most likely to conceive on the 13th day of the cycle.

**c)** This is a point of inflection.

Rate of change of temperature is $T'(t)$. For this quantity to be a maximum, we need its derivative $T''(t) = 0$. This is a necessary condition for an inflection point.

**Chapter 3 Section 3 Question 13 Page 175**

**a)** $f''(x) = x^2(x-2)$

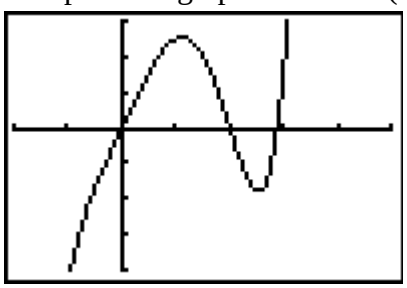
$$x^2(x-2) = 0$$

$$x = 0,\ x = 2$$

**b)** These values divide the domain into three intervals. Test the value of $f''(x)$ in each interval and summarize in a table.

| | $x<0$ | $0<x<2$ | $x>2$ |
|---|---|---|---|
| Test value | $-1$ | $1$ | $3$ |
| $f''(x)$ | $f''(-1)=-3$ Negative | $f''(1)=-1$ Negative | $f''(3)=9$ Positive |
| $f(x)$ | Concave down | Concave down | Concave up |

**c)** The graph start concave down, become straight at $x=0$, then continue concave down to $x=2$. This point (2, 1) will be a point of inflection, after which the curve will continue concave up. One possible graph is shown. (Its equation is $y=3x^5-10x^4+32x$).


(Yscl: 10)

**Chapter 3 Section 3 Question 14 Page 175**

There is one point of inflection on the graph of $f(x)$.
For a point of inflection for $f(x)$, we need the slope of $f'(x)$ to change from negative to positive (or vice versa). This can only occur if the slope of $f'(x)$ is zero, which occurs exactly once on this graph of $f'(x)$.

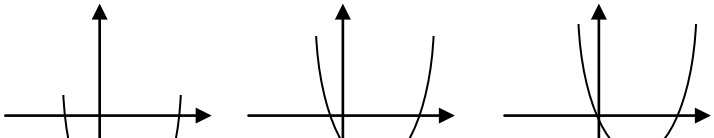
**Chapter 3 Section 3 Question 15 Page 175**

A quartic function $f(x)$ will have a cubic first derivative $f'(x)$ and a quadratic second derivative $f''(x)$.

For a point of inflection, the second derivative must change sign (i.e., positive to negative or negative to positive).
Since $f''(x)$ is quadratic, it can have 0, 1, or 2 zeros.
If $f''(x)$ has no zeros, it is always positive or always negative and does not change sign. There are no points of inflection in this case.

If $f''(x)$ has one zero, it must be a 'double' zero (two identical real roots). In this case, $f''(x)$ is always positive or zero, or negative or zero. In either case there is no change of sign and there are no points of inflection.
If $f''(x)$ has two zeros, it must go from positive to negative to positive (or the reverse) to create the two distinct zeros. Clearly there are two changes of sign in this case and consequently, two points of inflection.





The diagrams below show the three possibilities for $f''(x)$

**Chapter 3 Section 3 Question 16 Page 175**

$$f(x) = ax^3 + bx^2 + cx + d$$
$$f'(x) = 3ax^2 + 2bx + c$$
$$f''(x) = 6ax + 2b$$

Since (0, 2) is on the curve,

$$a(0)^3 + b(0)^2 + c(0) + d = 2$$
$$d = 2$$

Since (0, 2) is a point of inflection,

$$6a(0) + 2b = 0$$
$$b = 0$$

Since $x = 2$ is a critical point,

$$3a(2)^2 + 2(0)(2) + c = 0$$
$$c = -12a$$

Since (2, 6) is on the curve,

$$a(2)^3 + (0)(2)^2 + c(2) + (2) = 6$$
$$8a + 2(-12a) = 4$$
$$-16a = 4$$
$$a = -\frac{1}{4}$$

$$c = -12\left(-\frac{1}{4}\right)$$
$$= 3$$

The required function is $f(x) = -\frac{1}{4}x^3 + 3x + 2$.

**b)** A cubic function has range $y \in \mathbb{R}$. The curve cannot just have a local maximum. On one side of the maximum point it must descend to a minimum point and then increase again to satisfy the required range.

*Alternative explanation.*

In question 19 of section 3.2 it was proved that a cubic function has either two critical points or none. Since given one critical point is given, there must be a second one as well. Since intervals of increase/decrease alternate between critical points, the second critical point must be a local minimum.

**Chapter 3 Section 3 Question 17 Page 175**

**a)** Since $f''(x) = 2$ for $x \in \mathbb{R}$, $f'(x)$ must be a linear function and $f(x)$ must be a quadratic function.

**b)** If $f(x) = x^n$, $n \in \mathbb{N}$, then the degree of the function must be odd of degree at least 3 since there are two intervals of opposite concavity. Some examples: $f(x) = x^5$, $f(x) = x^3$, $f(x) = x^{11}$. Note that not all cubic functions work since $f(x) = x^3 - 2x$ does not.

**c) S**ince there are two intervals of opposite concavity, the degree of the function must be odd of degree at least 3.

**d) S**ince there are three intervals of alternating concavity, the degree of the function must be even of degree at least 4.

**e) S**ince there are two intervals of opposite concavity, the degree of the function must be odd of degree at least 3.

**f) S**ince there are three intervals of alternating concavity, the degree of the function must be even of degree at least 4.

**Chapter 3 Section 3 Question 18 Page 175**

E is the correct answer.

$f'(a)$ and $f''(a)$ do not necessarily exist.
Therefore, A and B are not necessarily true.

C is not true because *f* is not necessarily strictly increasing when $x < a$ and strictly decreasing when $x > a$.

D is not true if *f* is a function which is not differentiable everywhere.
Such a function exists. For example,

$$f(x) = \begin{cases} 1 & \text{if } x = a \\ 0 & \text{if } x \neq a \text{ is rational} \\ -1 & \text{if } x \neq a \text{ is irrational} \end{cases}$$

E is the definition of the local maximum.

**Chapter 3 Section 3 Question 19 Page 175**

E is the correct answer.

Let $f(x) = x^4$.

Then $f'(0) = 0 = f''(0)$ and $f$ has a local minimum at $x = 0$.

Let $f(x) = -x^4$.
Then $f'(0) = 0 = f''(0)$ and $f$ has a local maximum at $x = 0$.

Let $f(x) = x^3$.
Then $f'(0) = 0 = f''(0)$ and $f$ has a point of inflection at $x = 0$.

So $f'(a) = 0 = f''(a)$ can happen when there is a local maximum, a local minimum, or a point of inflection at $x = a$.

**Chapter 3 Section 4 Simple Rational Functions**

**Chapter 3 Section 4 Question 1 Page 183**

Vertical asymptotes only occur where the denominator in a rational function equals zero.

**a)** $x = 5$

**b)** $x^2 - 4 = 0$

$x^2 = 4$

$x = 2$ or $x = -2$

**c)** $x^2 + 5 = 0$

$x^2 = -5$

There are no vertical asymptotes since this equation has no real solutions.

**d)** $x^2 - 3x + 2 = 0$

$(x-2)(x-1) = 0$

$x = 2$ or $x = 1$

**e)** $x^2 + 2x - 4 = 0$

$$x = \frac{-2 \pm \sqrt{2^2 - 4(1)(\check{G}4)}}{2}$$

$$x = \frac{-2 \pm \sqrt{20}}{2}$$

$x = -1 + \sqrt{5}$ or $x = -1 - \sqrt{5}$

**f)** $x = 0$

**g)** $x^4 + 8 = 0$

$x^4 = -8$

There are no vertical asymptotes since this equation has no real solutions.

**h)** $x^2 - 6x + 9 = 0$

$(x-3)(x-3) = 0$

$x = 3$

**Chapter 3 Section 4 Question 2 Page 183**

**a)** Since $\frac{4.5}{4.5-5}=-9$, $\lim_{x\to 5^-} f(x)=-\infty$.

Since $\frac{5.5}{5.5-5}=+11$, $\lim_{x\to 5^+} f(x)=\infty$.

**b)** Since $\frac{1.9+3}{(1.9)^2-4}=-13$, $\lim_{x\to 2^-} f(x)=-\infty$.

Since $\frac{2.1+3}{(2.1)^2-4}=12$, $\lim_{x\to 2^+} f(x)=\infty$.

Since $\frac{-2.01+3}{(-2.01)^2-4}=25$, $\lim_{x\to -2^-} f(x)=\infty$.

Since $\frac{-1.99+3}{(-1.99)^2-4}=-25$, $\lim_{x\to -2^+} f(x)=-\infty$.

**c)** No asymptotes.

**d)** Since $\frac{(1.9)^2}{(1.9)^2-3(1.9)+2}=-40$, $\lim_{x\to 2^-} f(x)=-\infty$.

Since $\frac{(2.1)^2}{(2.1)^2-3(2.1)+2}=40$, $\lim_{x\to 2^+} f(x)=\infty$.

Since $\frac{(0.99)^2}{(0.99)^2-3(0.99)+2}=97$, $\lim_{x\to 1^-} f(x)=\infty$.

Since $\frac{(1.01)^2}{(1.01)^2-3(1.01)+2}=-103$, $\lim_{x\to 1^+} f(x)=-\infty$.

**e)** Since $\frac{1.23-5}{(1.23)^2+2(1.23)-4}=139$, $\lim_{x\to\left(-1+\sqrt{5}\right)^-} f(x)=\infty$.

Since $\frac{1.24-5}{(1.24)^2+2(1.24)-4}=-213$, $\lim_{x\to\left(-1+\sqrt{5}\right)^+} f(x)=-\infty$.

Since $\frac{(-3.24)-5}{(-3.24)^2+2(-3.24)-4}=-468$, $\lim_{x\to\left(-1-\sqrt{5}\right)^-} f(x)=-\infty$.

Since $\frac{(-3.23)-5}{(-3.23)^2+2(-3.23)-4}=304$, $\lim_{x\to\left(-1-\sqrt{5}\right)^+} f(x)=\infty$.

**f)** Since $2(-0.01)+\dfrac{1}{(-0.01)}=-100$, $\lim\limits_{x\to 0^-} f(x)=-\infty$.

Since $2(0.01)+\dfrac{1}{0.01}=100$, $\lim\limits_{x\to 0^+} f(x)=\infty$.

**g)** No asymptotes.

**h)** Since $\dfrac{2(2.99)-3}{(2.99)^2-6(2.99)+9}=29\ 800$, $\lim\limits_{x\to 3^-} f(x)=\infty$.

Since $\dfrac{2(3.01)-3}{(3.01)^2-6(3.01)+9}=30\ 200$, $\lim\limits_{x\to 3^+} f(x)=\infty$.

**Chapter 3 Section 4 Question 3 Page 183**

**a)**
$$\begin{aligned} y &= \frac{1}{x^2} \\ &= x^{-2} \\ y' &= -2x^{-3} \\ &= -\frac{2}{x^3} \end{aligned}$$

Critical points occur if the derivative is zero or undefined. Here, $x=0$ is the only possibility, but the function is not defined at $x=0$. Therefore, there are no local extrema.

**b)**
$$\begin{aligned} f(x) &= \frac{2}{x+3} \\ &= 2(x+3)^{-1} \\ f'(x) &= -2(x+3)^{-2} \\ &= -\frac{2}{(x+3)^2} \end{aligned}$$

Critical points occur if the derivative is zero or undefined. Here, $x=-3$ is the only possibility, but the function is not defined at $x=-3$. Therefore, there are no local extrema.

**c)**
$$\begin{aligned} g(x) &= \frac{x}{x-4} \\ &= x(x-4)^{-1} \\ g'(x) &= 1(x-4)^{-1}+x(-1)(x-4)^{-2}(1) \\ &= (x-4)^{-2}((x-4)-x) \\ &= \frac{-4}{(x-4)^2} \end{aligned}$$

Critical points occur if the derivative is zero or undefined. Here, $x=4$ is the only possibility, but the function is not defined at $x=4$. Therefore, there are no local extrema.

**d)** $h(x) = \dfrac{-3}{(x-2)^2}$

$= -3(x-2)^{-2}$

$h'(x) = 6(x-2)^{-3}$

$= \dfrac{6}{(x-2)^3}$

Critical points occur if the derivative is zero or undefined. Here, $x = 2$ is the only possibility, but the function is not defined at $x = 2$. Therefore, there are no local extrema.

**e)** $y = \dfrac{x}{x^2-1}$

$= x(x^2-1)^{-1}$

$y' = 1(x^2-1)^{-1} + x(\check{G}1)(x^2-1)^{-2}(2x)$

$= (x^2-1)^{-2}(x^2-1-2x^2)$

$= -\dfrac{x^2+1}{(x^2-1)^2}$

Critical points occur if the derivative is zero or undefined. Here, $x = 1$ is the only possibility, but the function is not defined at $x = 1$. Therefore, there are no local extrema.

**f)** $t(x) = \dfrac{2x}{3x^2+12x}$

$= 2x(3x^2+12x)^{-1}$

$t'(x) = 2(3x^2+12x)^{-1} + 2x(\check{G}1)(3x^2+12x)^{-2}(6x+12)$

$= (3x^2+12x)^{-2}(6x^2+24x-12x^2-24x)$

$= \dfrac{-6x^2}{(3x^2+12x)^2}$

Critical points occur if the derivative is zero or undefined. Here, $x = 0, -4$ is the only possibility, but the function is not defined at $x = 0, -4$. Therefore there are no local extrema.

**Chapter 3 Section 4 Question 4 Page 183**

**a)** The two functions are both rational functions having only one vertical asymptote. The graph of *f* is the graph of *g* translated one unit left and reflected in the *x*-axis.

**b)** $f(x) = \dfrac{-2}{(x+1)^2}$

$= -2(x+1)^{-2}$

$f'(x) = 4(x+1)^{-3}$

$= \dfrac{4}{(x+1)^3}$

There are no critical points. The intervals of increase/decrease are separated by the asymptote value, $x = -1$.

Use a table to show increasing and decreasing intervals for the function.

| | $x < -1$ | $x = -1$ | $x > -1$ |
|---|---|---|---|
| **Test Value** | $-2$ | | $0$ |
| $f'(x)$ | $f'(-2) = -4$ Negative | Undefined | $f'(0) = 4$ Positive |
| $f(t)$ | Decreasing ↘ | Vertical Asymptote | Increasing ↗ |

**c)** $f(x) = \dfrac{-2}{(x+1)^2}$

$$f'(x) = 4(x+1)^{-3}$$

$$f''(x) = -12(x+1)^{-4}$$

$$= \frac{-12}{(x+1)^4}$$

Concavity changes only at the asymptotes.

| | $x < -1$ | $x = -1$ | $x > -1$ |
|---|---|---|---|
| **Test Value** | $-2$ | | $0$ |
| $f''(x)$ | $f''(-2) = -12$ Negative | Undefined | $f''(0) = -12$ Negative |
| $f(t)$ | Concave down ⌒ | Vertical Asymptote | Concave down ⌒ |

**Chapter 3 Section 4 Question 5 Page 183**

**a)** $x^2 - 4 = 0$

$x = -2,\ x = 2$

The equations of the vertical asymptotes are $x = -2$ and $x = 2$.

**b)** Find the critical points.

$$h(x) = \frac{1}{x^2 - 4}$$

$$= (x^2 - 4)^{-1}$$

$$h'(x) = -(x^2 - 4)^{-2}(2x)$$

$$= -\frac{2x}{(x^2 - 4)^2}$$

There is a critical point at $x = 0$.

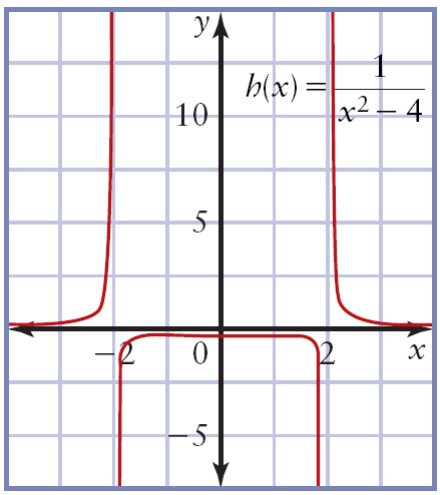
Consider the intervals defined by this point and the asymptotes.

| | $x < -2$ | $-2 < x < 0$ | $0 < x < 2$ | $x > 2$ |
|---|---|---|---|---|
| **Test Value** | $-3$ | $-1$ | $1$ | $3$ |
| $h'(x)$ | $h'(-3) = 0.24$ Positive | $h'(-1) = 0.22$ Positive | $h'(1) = -0.22$ Negative | $h'(3) = -0.24$ Negative |
| $h(t)$ | Increasing ↗ | Increasing ↗ | Decreasing ↘ | Decreasing ↘ |

**c)** Since the curve is increasing to the left of $x = -2$, the left hand limit must be $\infty$.
Since the curve is increasing to the right of $x = -2$, the right hand limit must be $-\infty$.

Since the curve is decreasing to the left of $x = 2$, the left hand limit must be $-\infty$.
Since the curve is decreasing to the right of $x = 2$, the right hand limit must be $\infty$.

**d)**




**Chapter 3 Section 4** **Question 6 Page 183**

**a)** $C(p) = \dfrac{75\,000}{100 - p}$

$$C(50) = \frac{75\,000}{50}$$
$$= 1500$$

The removal cost is \$1500.

**b)** Since $C(99.9) = 750\ 000$, $\lim\limits_{p \to 100^-} C(p) = \infty$.

**c)** Since the cost increases without bound as the percentage of contaminants removed approaches 100, it is clear that there is not enough money available to achieve 100% removal.

**Chapter 3 Section 4 Question 7 Page 183**

If the curve is concave down for $x > 2$ and must approach the asymptote asymptotically, it will look like a parabola opening down in this interval. Therefore, on the right side of the asymptote it must approach $-\infty$. In symbols, $\lim\limits_{x \to 2^+} f(x) = -\infty$.

**Chapter 3 Section 4 Question 8 Page 184**

**a)** $N(t) = 0.5t + \dfrac{5}{10t+1}$

$$\begin{aligned} N(0) &= 0.5(0) + \frac{5}{10(0)+1} \\ &= 5 \end{aligned}$$

There were 5 units of pollutant in the river when the clean-up began.

**b)** To find the minimum, look for critical points.

$$\begin{aligned} N(t) &= 0.5t + \frac{5}{10t+1} \\ &= 0.5t + 5(10t+1)^{-1} \end{aligned}$$

$$\begin{aligned} N'(t) &= 0.5 - 510t + 1^{-2}(10) \\ &= 0.5 - \frac{50}{10t+1^2} \end{aligned}$$

$$\begin{aligned} 0.5 - \frac{50}{10t+1^2} &= 0 \\ 10t + 1^2 &= 100 \\ 10t + 1 &= \pm 10 \end{aligned}$$

$t = -1.1,\ t = 0.9$

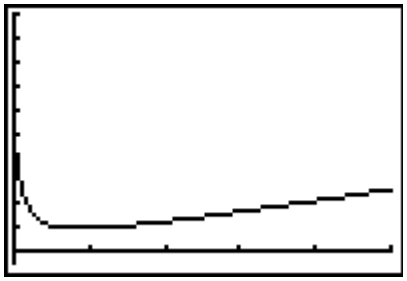
Since time must be positive, there is only one critical point at $t = 0.9$.

Since $N'(0.5) = -0.9$, $N$ is decreasing before the critical point.
Since $N'(1) = 0.1$, $N$ is increasing after the critical point.

The quantity of pollutant is a minimum after about 0.9 years (about 11 months).

Since $\frac{N(0.9)}{N(0)} = \frac{0.95}{5} = \frac{19}{100}$, the fraction of pollutant remaining is about $\frac{1}{5}$.

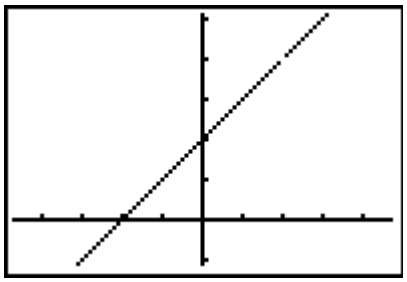
**c)** Since the pollutants started to increase again after this time, it may be possible that the environmental group stopped working on the project at this time. It is also possible that the source of pollution had begun to reintroduce additional pollutants into the river at this time.

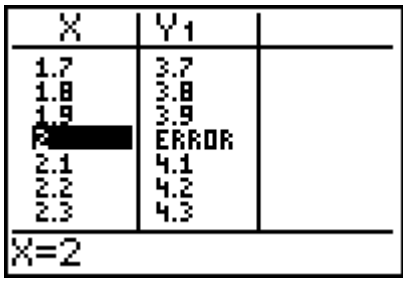


**Chapter 3 Section 4 Question 9 Page 184**

Solutions for Achievement Checks are shown in the Teacher's Resource.

**Chapter 3 Section 4 Question 10 Page 184**






**a)** The limit of the function values close to $x = 2$ is not $\infty$ or $-\infty$, which is necessary for an asymptote. Examining the function closely, we can see that the numerator and denominator of the fraction share a common factor of $x - 2$. If this divided out, the function becomes $f(x) = x + 2,\ x \neq 2$

**b)** No. the function is not defined at $x = 2$. In fact, $f(2) = \frac{0}{0}$ which is an undefined quantity.

**Chapter 3 Section 4 Question 11 Page 184**

$$f(x)=\frac{ax}{bx+c}=ax(bx+c)^{-1}$$

$$f'(x)=a(bx+c)^{-1}+ax(\check{G}1)(bx+c)^{-2}(b) \quad \text{Product rule}$$

$$=(bx+c)^{-2}(a(bx+c)-abx)$$

$$=\frac{c}{(bx+c)^2}$$

Turning points (critical points) occur only if the derivative is zero or undefined.
The derivative cannot be zero since the numerator is a non-zero constant.
The denominator cannot be zero, which would lead to an undefined derivative, since a quantity squared is always greater than zero.
Therefore there are no turning points possible.

**Chapter 3 Section 4 Question 12 Page 184**

The function $f(x)=\frac{1}{(x-2)(x+1)}$ will have asymptotes at $x=2$ and $x=-1$.

Since this function is below the $x$-axis when $x=1$, add a constant to the function to guarantee the $x$-intercept of 1.

$$\frac{1}{(1-2)(1+1)}+k=0$$

$$-\frac{1}{2}+k=0$$

$$k=\frac{1}{2}$$

One possible function is $f(x)=\frac{1}{(x-2)(x+1)}+\frac{1}{2}$.

*Alternative solution.*
Begin as above but introduce a factor in the numerator that will have a value of zero when $x=1$.

Another possible function is $f(x)=\frac{x-1}{(x-2)(x+1)}$.

**Chapter 3 Section 4 Question 13 Page 184**

B is the correct answer.

$$y = \frac{(x+1)^2}{2x^2+5x+3}$$
$$= \frac{(x+1)^2}{(x+1)(2x+3)}, \text{ where } x \neq -1$$
$$= \frac{x+1}{2x+3}$$

Therefore $x = -1$ cannot be a vertical asymptote as $-1$ is not in the domain of the function.
Similarly, $-1$ cannot be an $x$-intercept.
$\lim_{x\to\infty} \frac{x+1}{2x+3} = \frac{1}{2}$ implies that $y = \frac{1}{2}$ is a horizontal asymptote.

**Chapter 3 Section 4 Question 14 Page 184**

C is the correct answer.

A. Since $x^n + 1 \neq 0$,
$$y = \frac{x^{2n}-1}{x^{2n}+x^n}$$
$$= \frac{(x^n-1)(x^n+1)}{x^n(x^n+1)}$$
$$= \frac{x^n-1}{x^n}$$
There is a vertical asymptote $x = 0$ and a horizontal asymptote $y = 1$.

B. $y = \frac{x^{2n}+1}{x^n+1}$ has a vertical asymptote $x = \sqrt[n]{-1}$ when $n$ is odd.

C. Since $x^n + 1 \neq 0$,
$$y = \frac{x^{2n}-1}{x^n+1}$$
$$= \frac{(x^n-1)(x^n+1)}{x^n+1}$$
$$= x^n - 1$$
This is a polynomial function and clearly does not have an asymptote.

D. For $x^n - 1 \neq 0$,

$$\begin{aligned} y &= \frac{x^{2n} - x^n}{x^{2n} + x^n - 2} \\ &= \frac{x^n(x^n - 1)}{(x^n + 2)(x^n - 1)} \\ &= \frac{x^n}{x^n + 2} \\ &= \frac{x^n + 2 - 2}{x^n + 2} \\ &= 1 - \frac{2}{x^n + 2} \end{aligned}$$

There is a vertical asymptote $x = \sqrt[n]{-2}$ when $n$ is odd and a horizontal asymptote $y = 1$.

E. $$\begin{aligned} y &= \frac{x^{2n+1} + x + 1}{x^{2n} + 1} \\ &= \frac{x(x^{2n} + 1) + 1}{x^{2n} + 1} \\ &= x + \frac{1}{x^{2n} + 1} \end{aligned}$$

This function has a slant asymptote $y = x$.

**Chapter 3 Section 5** **Putting It All Together**

**Chapter 3 Section 5** **Question 1 Page 192**

**a)** $f(x) = x^3 - 6x$

$f'(x) = 3x^2 - 6$

$f''(x) = 6x$

Find the critical points.

$3x^2 - 6 = 0$

$x^2 = 2$

$x = \pm\sqrt{2}$

Use the second derivative test for extrema.

$f''\left(\sqrt{2}\right) = 6\left(\sqrt{2}\right)$

B8

$f''\left(-\sqrt{2}\right) = 6\left(-\sqrt{2}\right)$

B–8

Therefore, $\left(-\sqrt{2}, 4\sqrt{2}\right)$ is a local maximum and $\left(\sqrt{2}, -4\sqrt{2}\right)$ is a local minimum.

**b)** $g(x) = -x^4 + 2x^2$

$g'(x) = -4x^3 + 4x$

$g''(x) = -12x^2 + 4$

Find the critical points.

$-4x^3 + 4x = 0$

$-4x(x^2 - 1) = 0$

$-4x(x+1)(x-1) = 0$

$x = 0,\ x = 1,\ x = -1$

Use the second derivative test for extrema.

$g''(-1) = -12(-1)^2 + 4$

$= -8$

$g''(1) = -12(1)^2 + 4$

$= -8$

$g''(0) = -12(0)^2 + 4$

$= 4$

Therefore, (–1, 1) and (1, 1) are local maximum points and (0, 0) is a local minimum.

**c)** $f(x) = -x^3 + 3x - 2$

$f'(x) = -3x^2 + 3$

$f''(x) = -6x$

Find the critical points.

$-3x^2 + 3 = 0$

$x^2 = 1$

$x = 1, x = -1$

Use the second derivative test for extrema.

$f''(-1) = -6(-1)$

$= 6$

$f''(1) = -6(-1)$

$= -6$

Therefore, (1, 0) is a local maximum and (–1, –4) is a local minimum.

**d)** $h(x) = 2x^2 + 4x + 5$

$h'(x) = 4x + 4$

$h''(x) = 4$

Find the critical points.

$4x + 4 = 0$

$x = -1$

Use the second derivative test for extrema.

$h''(-1) = 4$

Therefore, (–1, 3) is a local minimum.

**Chapter 3 Section 5 Question 2 Page 192**

**a)** $f(x) = 2x^3 - 4x^2$

$f'(x) = 6x^2 - 8x$

$f''(x) = 12x - 8$

For points of inflection, let $f''(x) = 0$.

$12x - 8 = 0$

$x = \frac{2}{3}$

Test if $f''(x)$ changes sign.

$f''(0) = -8$

$f''(1) = 4$

Since there is a sign change, $\left(\frac{2}{3}, -\frac{32}{27}\right)$ is a point of inflection.

**b)** $f(x) = x^4 - 6x^2$

$f'(x) = 4x^3 - 12x$

$f''(x) = 12x^2 - 12$

For points of inflection, let $f''(x) = 0$.

$12x^2 - 12 = 0$

$x^2 = 1$

$x = 1, x = -1$

Test if $f''(x)$ changes sign.

For $x = -1$,

$f''(-2) = 36$

$f''(0) = -12$

For $x = 1$,

$f''(0) = -12$

$f''(2) = 36$

Since there is a sign change in each case, (–1, –5) and (1, –5) are points of inflection.

**c)** $f(x) = x^5 - 30x^3$

$f'(x) = 5x^4 - 90x^2$

$f''(x) = 20x^3 - 180x$

For points of inflection, let $f''(x) = 0$.

$20x^3 - 180x = 0$

$20x(x^2 - 9) = 1$

$x = 0,\ x = 3, x = -3$

Test if $f''(x)$ changes sign.

For $x = -3$,

$f''(-4) = -560$

$f''(-1) = 160$

For $x = 3$,
$f''(1) = -160$
$f''(4) = 560$

For $x = 0$,
$f''(-1) = 160$
$f''(1) = -160$

Since there is a sign change in each case, (–3, 567), (3, –567), and (0, 0) are points of inflection.

**d)** $f(x) = 3x^5 - 5x^4 - 40x^3 + 120x^2$
$f'(x) = 15x^4 - 20x^3 - 120x^2 + 240x$
$f''(x) = 60x^3 - 60x^2 - 240x + 240$

For points of inflection, let $f''(x) = 0$.

$$60x^3 - 60x^2 - 240x + 240 = 0$$
$$60x^2(x-1) - 240(x-1) = 0$$
$$(x-1)(60)(x^2-4) = 0$$

$x = 1,\ x = 2,\ x = -2$

Test if $f''(x)$ changes sign.
For $x = -2$,
$f''(-3) = -1200$
$f''(-1) = 360$

For $x = 2$,
$f''(1.5) = -52.5$
$f''(3) = 600$

For $x = 1$,
$f''(0) = 240$
$f''(1.5) = -52.5$

Since there is a sign change in each case, (–2, 624), (2, 176), and (0, 0) are points of inflection.

**Chapter 3 Section 5 Question 3 Page 192**

**a)** $f(x) = x^4 - 8x^3$

$-f(x) = -x^4 + 8x^3$

$f(-x) = (-x)^4 - 8(-x)^3$

$= x^4 + 8x^3$

The function is neither even nor odd.

**b)** The function is a polynomial. The domain is $\{x \in \mathbb{R}\}$.

**c)** For the $y$-intercept, let $x = 0$.

$f(0) = 0$

The $y$-intercept is 0.

For the $x$-intercept, let $y = 0$.

$x^4 - 8x^3 = 0$

$x^3(x - 8) = 0$

$x = 0,\ x = 8$

The $x$-intercepts are 0 and 8.

**d)** Find the critical points.

$f(x) = x^4 - 8x^3$

$f'(x) = 4x^3 - 24x^2$

$f''(x) = 12x^2 - 48x$

$4x^3 - 24x^2 = 0$

$4x^2(x - 6) = 0$

$x = 0,\ x = 6$

Use the second derivative test for extrema.

$f''(0) = 12(0)^2 - 48(0)$

$= 0$

$f''(-1) = 12(-1)^2 - 48(-1)$

$= 60$

$f''(1) = 12(1)^2 - 48(1)$

$= -36$

Note that for $x = 0$, check the sign of $f''(x)$ on either side of $x = 0$. Since the function is changing from concave up to concave down there, (0, 0) is a point of inflection.

$$f''(6) = 12(6)^2 - 48(6)$$
$$= 144$$

Therefore, (6, –432) is a local minimum.

The critical points create three intervals. Test the derivative in each interval.

| | $x < 0$ | $0 < x < 6$ | $x > 6$ |
|---|---|---|---|
| **Test Value** | –1 | 1 | 7 |
| $f'(x)$ | $f'(-1) = -28$ Negative | $f'(1) = -20$ Negative | $f'(7) = 196$ Positive |
| $f(x)$ | Decreasing | Decreasing | Increasing |

The function is decreasing for $x < 0$ and $0 < x < 6$ and is increasing for $x > 6$.

For concavity, we need to find all points of inflection. Let $f''(x) = 0$.

$$12x^2 - 48x = 0$$
$$12x(x - 4) = 0$$

$$x = 0,\ x = 4$$

Test for concavity in the intervals between the possible points of inflection.

| | $x < 0$ | $x = 0$ | $0 < x < 4$ | $x = 4$ | $x > 4$ |
|---|---|---|---|---|---|
| **Test value** | –1 | | 1 | | 5 |
| $f''(x)$ | $f''(-1) = 60$ Positive | 0 | $f''(1) = -36$ Negative | 0 | $f''(5) = 60$ Positive |
| $f(x)$ | Concave up | Point of Inflection (0, 0) | Concave down | Point of Inflection (4, –256) | Concave up |

**Chapter 3 Section 5 Question 4 Page 192**

**a)** Follow the six step plan.

$$f(x) = x^3$$
$$f'(x) = 3x^2$$
$$f''(x) = 6x$$

Step 1. The domain is $\{x \in \mathbb{R}\}$.

Step 2. $f(0) = 0$

The $y$-intercept is 0.

$$x^3 = 0$$
$$x = 0$$

The $x$-intercept is 0.

Step 3. Find the critical numbers.

$$3x^2 = 0$$
$$x = 0$$

Use the second derivative test to classify the critical point.

$$f''(0) = 6(0)$$
$$= 0 \qquad \text{Check further.}$$
$$f''(-1) = -6$$
$$f''(1) = 6$$

Step 3. ④ Therefore (0, 1) is a point of inflection and not an extremum.

Step 5. $f'(-1) = 3$
$f'(1) = 3$

Therefore, $f$ is increasing for $x < 0$ and $x > 0$.

From Step 3, $f$ is concave down for $x < 0$ and concave up for $x > 0$.

Step 6. Sketch the graph.

QuickTime™ and a
TIFF (LZW) decompressor
are needed to see this picture.

$(x \in [\check{G}4, 4],\ y \in [\check{G}4, 4])$

**b)** Follow the six step plan.

$$h(x) = x^5 + 20x^2 + 5$$
$$h'(x) = 5x^4 + 40x$$
$$h''(x) = 20x^3 + 40$$

Step 1. The domain is $\{x \in \circ\}$.

Step 2. $h(0) = 5$.
The $y$-intercept is 5.

$$x^5 + 20x^2 + 5 = 0$$

This equation is not factorable. Trial and error suggests there is an $x$-intercept of around $x = -3$.

Step 3. Find the critical numbers.

$$5x^4 + 40x = 0$$
$$5x(x^3 + 8) = 0$$
$$5x(x+2)(x^2 - 2x + 4) = 0 \qquad \text{The third factor has no real roots.}$$

$$x = 0,\ x = -2$$

Use the second derivative test to classify the critical points.

$$h''(0) = 20(0)^3 + 40$$
$$= 40$$
$$h''(-2) = 20(-2)^3 + 40$$
$$= -120$$

Therefore, (–2, 53) is a local maximum and (0, 5) is a local minimum.

Step 4. Find the possible points of inflection.

$$20x^3 + 40 = 0$$
$$x^3 + 2 = 0 \qquad \text{Factor the sum of cubes.}$$
$$\left(x + \sqrt[3]{2}\right)\left(x^2 - \sqrt[3]{2}x + \left(\sqrt[3]{2}\right)^2\right) = 0 \qquad \text{The second factor has no real roots.}$$

$$x = -\sqrt[3]{2}$$

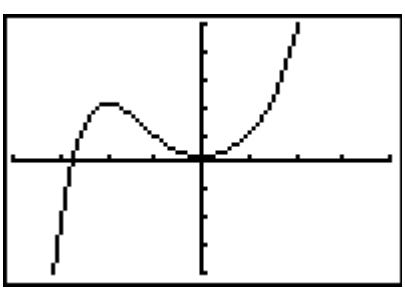
Test the intervals.

$$h''(-2) = 20(-2)^3 + 40$$
$$= -120$$
$$h''(0) = 20(0)^3 + 40$$
$$= 40$$

Therefore $\left(-\sqrt[3]{2}, 18\left(\sqrt[3]{2}\right)^2 + 5\right)$ or (–1.26, 33.57) is a point of inflection.

Step 5. From Step 3, $h$ is increasing for $x < -2$ and $x > 0$ and decreasing for $-2 < x < 0$.
From Step 4, $h$ is concave down for $x < -\sqrt[3]{2}$ and concave up for $x > -\sqrt[3]{2}$.

Step 6. Sketch the graph.



( $x \in [-4, 4]$, $y \in [-100, 125]$; Yscl:25 )

**c)** Follow the six step plan.

$$k(x) = \frac{1}{2}x^4 - 2x^3$$
$$k'(x) = 2x^3 - 6x^2$$
$$k''(x) = 6x^2 - 12x$$

Step 1. The domain is $\{x \in \mathbb{R}\}$.

Step 2. $k(0) = 0$

The $y$-intercept is 0.

$$\frac{1}{2}x^4 - 2x^3 = 0$$
$$x^3(x-4) = 0$$

$x = 0,\ x = 4$

The $x$-intercepts are 0 and 4.

Step 3. Find the critical numbers.

$$2x^3 - 6x^2 = 0$$
$$2x^2(x-3) = 0$$

$x = 0,\ x = 3$

Use the second derivative test to classify the critical points.

$$k''(0) = 6(0)^2 - 12(0)$$
$$= 0 \qquad \text{Check further.}$$
$$k''(-1) = 6(-1)^2 - 12(-1)$$
$$= 18$$
$$k''(1) = 6(1)^2 - 12(1)$$
$$= -6$$
$$k''(3) = 6(3)^2 - 12(3)$$
$$= 18$$

Therefore (3, –13.5) is a local minimum.

Step 4. Find the possible points of inflection.

$$6x^2 - 12x = 0$$
$$6x(x-2) = 0$$

$x = 0,\ x = 2$

Test the intervals.

$x = 0$ was tested already.

$k''(1) = -6$

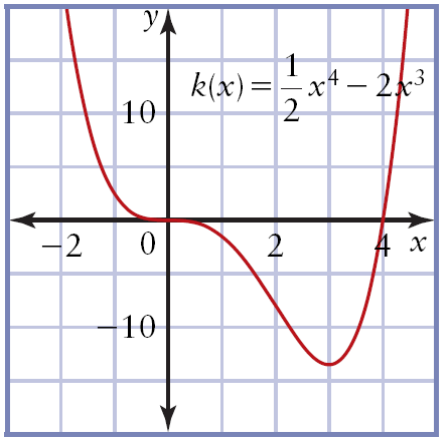
$k''(3) = 6(3)^2 - 12(3)$

$= 18$

Therefore (0, 0) and (2, –8) are points of inflection.

Step 5. From Step 3, $k$ is increasing for $x > 3$ and decreasing for $x < 0$ and $0 < x < 3$.
From Step 4, $k$ is concave down for $0 < x < 2$ and concave up for $x < 0$ and $x > 2$.

Step 6. Sketch the graph.




**d)** Follow the six step plan.

$b(x) = -(2x-1)(x^2 - x - 2)$

$= -2x^3 + 3x^2 + 3x - 2$

$b'(x) = -6x^2 + 6x + 3$

$b''(x) = -12x + 6$

Step 1. The domain is $\{x \in \mathbb{R}\}$.

Step 2. $b(0) = -2$

The $y$-intercept is –2.

$-(2x-1)(x^2 - x - 2) = 0$

$-(2x-1)(x-2)(x+1) = 0$

$x = \frac{1}{2},\ x = 2,\ x = -1$

The $x$-intercepts are –1, 0.5, and 2.

Step 3. Find the critical numbers.

$$-6x^2 + 6x + 3 = 0$$

$$-3(2x^2 - 2x - 1) = 0$$

$$x = \frac{2 \pm \sqrt{12}}{4}$$

$$= \frac{1 \pm \sqrt{3}}{2}$$

$$x \approx -0.4,\ x \approx 1.4$$

Use the second derivative test to classify the critical points.

$$b''\left(\frac{1-\sqrt{3}}{2}\right) = -12\left(\frac{1-\sqrt{3}}{2}\right) + 6$$

$$= 6\sqrt{3}$$

$$b''\left(\frac{1+\sqrt{3}}{2}\right) = -12\left(\frac{1+\sqrt{3}}{2}\right) + 6$$

$$= -6\sqrt{3}$$

Therefore, (–0.4, –2.6) is a local minimum and (1.4, 2.6) is a local maximum.

Step 4. Find the possible points of inflection.

$$-12x + 6 = 0$$

$$x = \frac{1}{2}$$

Test the intervals.

$$b''(0) = -12(0) + 6$$

$$= 6$$

$$b''(1) = -12(1) + 6$$

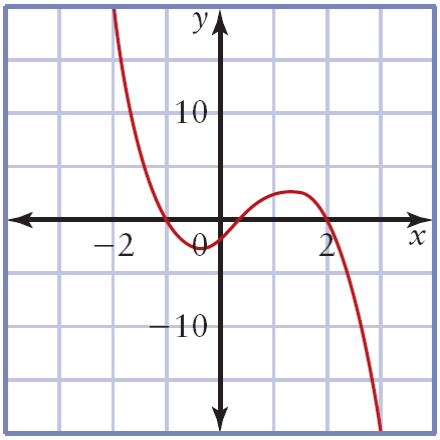
$$= -6$$

Therefore, (0.5, 0) is a point of inflection.

Step 5. From Step 3, $b$ is increasing for $\frac{1-\sqrt{3}}{2} < x < \frac{1+\sqrt{3}}{2}$ and decreasing for $x < \frac{1-\sqrt{3}}{2}$ and $x > \frac{1+\sqrt{3}}{2}$.

From Step 4, $b$ is concave up for $x < \frac{1}{2}$ and concave down for $x > \frac{1}{2}$.

Step 6. Sketch the graph.




**Chapter 3 Section 5 Question 5 Page 192**

$f(x) = 2x^3 - 3x^2 - 72x + 7$

**a)** The first derivative of *f* will be of degree 2. Setting the derivative to zero can yield at most two critical values. Therefore, the function has at most two local extrema.

**b)** The second derivative of *f* will be of degree 1. Setting this derivative equal to zero will yield one possible value for a point of inflection. Therefore, *f* can have at most one point of inflection.

**c)** $f'(x) = 6x^2 - 6x - 72$

$f''(x) = 12x - 6$

For critical values,

$$6x^2 - 6x - 72 = 0$$
$$x^2 - x - 12 = 0$$
$$(x-4)(x+3) = 0$$

$x = 4,\ x = -3$

Use the second derivative test to classify the critical points.

$$f''(-3) = 12(-3) - 6$$
$$= -42$$
$$f''(4) = 12(4) - 6$$
$$= 42$$

Therefore, (–3, 142) is a local maximum and (4, –201) is a local minimum.

Find the possible points of inflection.

$$12x - 6 = 0$$
$$x = \frac{1}{2}$$

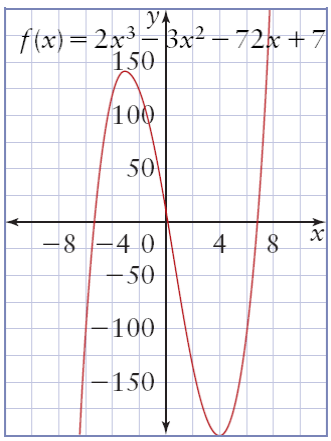
Test the intervals.

$$f''(0) = 12(0) - 6$$
$$= -6$$
$$f''(1) = 12(1) - 6$$
$$= 6$$

Therefore, (0.5, –29.5) is a point of inflection.

Therefore, *f* is increasing for $x < -3$ and $x > 4$ and decreasing for $-3 < x < 4$.
Also *f* is concave down for $x < 0.5$ and concave up for $x > 0.5$.

**d)**




**e)** This function has the maximum number of critical points and points of inflection predicted in parts a) and b).

**Chapter 3 Section 5 Question 6 Page 192**

**i)** $h(x) = 3x^2 - 27$

**a)** The first derivative of *h* will be of degree 1. Setting the derivative to zero will yield one critical value. Therefore, the function has one local extremum.

**b)** The second derivative of *h* will be of degree 0. There will be no points of inflection.

**c)** $h'(x) = 6x$
$h''(x) = 6$

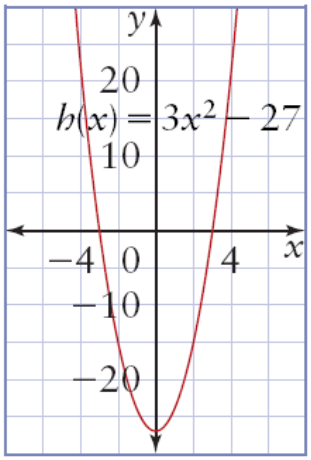
For critical values,
$6x = 0$
$x = 0$

Use the second derivative test to classify the critical points.
$h''(0) = 6$

Therefore, (0, –27) is a local minimum.
There are no points of inflection and the second derivative is always positive.

Therefore $h$ is decreasing for $x < 0$ and increasing for $x > 0$.
Also $h$ is always concave up.

**d)**




**e)** This function has the predicted number of critical points and points of inflection.

**ii)** $t(x) = x^5 - 2x^4 + 3$

**a)** The first derivative of $t$ will be of degree 4. Setting the derivative to zero can yield at most four critical values. Therefore, the function has at most three local extrema.

**b)** The second derivative of $t$ will be of degree 3. Setting this derivative equal to zero will yield at most three possible values for points of inflection. Therefore $t$ can have at most three points of inflection.

**c)** $t'(x) = 5x^4 - 8x^3$

$t''(x) = 20x^3 - 24x^2$

For critical values,

$5x^4 - 8x^3 = 0$

$x^3(5x - 8) = 0$

$x = 0,\ x = 1.6$

Use the second derivative test to classify the critical points.

$$t''(0) = 20(0)^3 - 24(0)^2$$
$$= 0$$

Check further.

$$t''(-1) = 20(-1)^3 - 24(-1)^2$$
$$= -4$$
$$t''(1) = 20(1)^3 - 24(1)^2$$
$$= -4$$
$$t''(1.6) = 20(1.6)^3 - 24(1.6)^2$$
$$= -20.48$$

Therefore (0, 3) is a local maximum and (1.6, 0.37856) is a local minimum.

Find the possible points of inflection.

$$20x^3 - 24x^2 = 0$$
$$4x^2(5x - 6) = 0$$

$$x = 0,\ x = 1.2$$

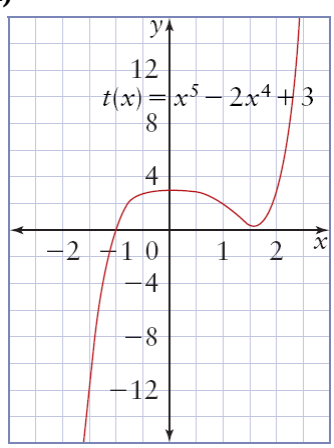
Test the intervals.

$$t''(-1) = 20(-1)^3 - 24(-1)^2$$
$$= -44$$
$$t''(1) = 20(1)^3 - 24(1)^2$$
$$= -4$$
$$t''(2) = 20(2)^3 - 24(2)^2$$
$$= 64$$

Therefore, (1.2, 1.3411) is the only point of inflection.

Therefore $t$ is increasing for $x < 0$ and $x > 1.6$ and decreasing for $0 < x < 1.6$.
Also $t$ is concave down for $x < 1.2$ and concave up for $x > 1.2$

**d)**



**e)** This function has less than the maximum number of predicted number of critical points and points of inflection.

**iii)** $g(x) = x^4 - 8x^2 + 16$

**a)** The first derivative of $g$ will be of degree 3. Setting the derivative to zero can yield at most three critical values. Therefore, the function has at most 3 local extrema.

**b)** The second derivative of $g$ will be of degree 2. Setting this derivative equal to zero will yield at most two possible values for points of inflection. Therefore $g$ can have at most two points of inflection.

**c)** $g'(x) = 4x^3 - 16x$

$g''(x) = 12x^2 - 16$

For critical values,

$4x^3 - 16x = 0$

$4x(x^2 - 4) = 0$

$x = 0,\ x = 2,\ x - 2$

Use the second derivative test to classify the critical points.

$$\begin{aligned} g''(-2) &= 12(-2)^2 - 16 \\ &= 32 \\ g''(0) &= 12(0)^2 - 16 \\ &= -16 \\ g''(2) &= 12(2)^2 - 16 \\ &= 32 \end{aligned}$$

Therefore, (–2, 0) and (2, 0) are local minimums and (0, 16) is a local maximum.

Find the possible points of inflection.

$$\begin{aligned} 12x^2 - 16 &= 0 \\ x^2 &= \frac{4}{3} \\ x &= \pm\frac{2}{\sqrt{3}} \end{aligned}$$

$x \approx -1.15,\ x \approx 1.15$

Test the intervals.

$g''(-2) = 32$

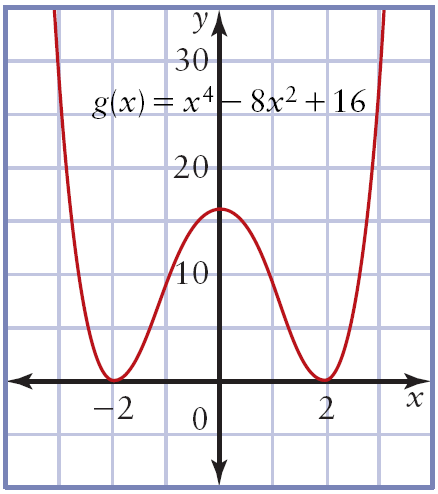
$g''(0) = -16$

$g''(2) = 32$

Therefore, (–1.15, 7.11) and (1.15, 7.11) are points of inflection.

Therefore, $g$ is decreasing for $x < -2$ and $0 < x < 2$ and increasing for $-2 < x < 0$ and $x > 2$. Also $g$ is concave up for $x < -1.15$ and $x > 1.15$ and concave down for $-1.15 < x < 1.15$.

**d)**




**e)** This function has the predicted maximum number of critical points and points of inflection.

**iv)** $k(x) = -2x^4 + 16x^2 - 12$

**a)** The first derivative of $k$ will be of degree 3. Setting the derivative to zero can yield at most three critical values. Therefore, the function has at most three local extrema.

**b)** The second derivative of $k$ will be of degree 2. Setting this derivative equal to zero will yield at most two possible values for points of inflection. Therefore, $k$ can have at most two points of inflection.

**c)** $k'(x) = -8x^3 + 32x$

$k''(x) = -24x^2 + 32$

For critical values,

$-8x^3 + 32x = 0$

$-8x(x^2 - 4) = 0$

$x = 0,\ x = 2,\ x = -2$

Use the second derivative test to classify the critical point.

$$k''(0) = -24(0)^2 + 32$$
$$= +32$$
$$k''(-2) = -24(-2)^2 + 32$$
$$= -64$$
$$k''(2) = -24(2)^2 + 32$$
$$= -64$$

Therefore (0, –12) is a local minimum and (–2, 20) and (2, 20) are local maximum points.

Find the possible points of inflection.

$$-24x^2 + 32 = 0$$
$$-8(3x^2 - 4) = 0$$

$$x = \sqrt{\frac{4}{3}},\ x = -\sqrt{\frac{4}{3}}$$
$$x \approx 1.15,\ x \approx -1.15$$

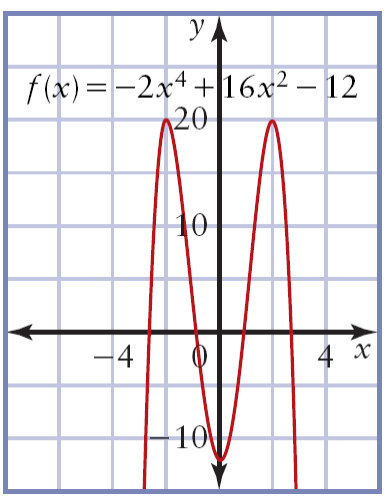
Test the intervals. (Note this was performed earlier.)

$$k''(-2) = -64$$
$$k''(0) = 32$$
$$k''(2) = -64$$

Therefore, (–1.15, 5.67) and (1.15, 5.67) are points of inflection.

Therefore, $k$ is increasing for $x < -2$ and $0 < x < 2$ and decreasing for $-2 < x < 0$ and $x > 2$.
Also $k$ is concave down for $x < -1.15$ and $x > 1.15$ and concave up for $-1.15 < x < 1.15$.

**d)**




**e)** This function has the predicted maximum number of critical points and points of inflection.

**Chapter 3 Section 5 Question 7 Page 192**

**a)** The maximum possible number of local extrema is one less than the degree of the polynomial function.

**b)** The number may be less that the maximum if one of the roots of the first derivative function is a 'double' root. In this case, the function is increasing (or decreasing) on both sides of the root and so there is no local extremum present.

**Chapter 3 Section 5 Question 8 Page 192**

Such a polynomial can have a maximum of 4 points of inflection. It is possible for it to have zero points of inflection. A simple example is $f(x) = x^6$.

**Chapter 3 Section 5 Question 9 Page 192**

**a)** Follow the six step plan.

$k(x) = 3x^3 + 7x^2 + 3x - 1$

$k'(x) = 9x^2 + 14x + 3$

$k''(x) = 18x + 14$

Step 1. Since this is a polynomial function, the domain is $\{x \in \mathbb{R}\}$.

Step 2. $k(0) = -1$

The $y$-intercept is $-1$.

For $x$-intercepts, let $y = 0$.

$3x^3 + 7x^2 + 3x - 1 = 0$ Use the factor theorem.

$(x+1)(3x^2 + 4x - 1) = 0$

$x = -1,\ x = \dfrac{-4 \pm \sqrt{28}}{6}$

$x = -1,\ x \approx 0.22,\ x \approx -1.55$

The $x$-intercepts are $-1$ and $\dfrac{-2 \pm \sqrt{7}}{3}$.

Step 3. Find the critical numbers.

$9x^2 + 14x + 3 = 0$

$x = \dfrac{-14 \pm \sqrt{88}}{18}$

$= \dfrac{-7 \pm \sqrt{22}}{9}$

$x \approx -1.3,\ x \approx -0.26$

Use the second derivative test to classify the critical points.

$k''(-1.3) = -9.4$

$k''(-0.26) = 9.32$

Therefore (–1.3, –0.34) is a local maximum and (–0.26, –1.36) is a local minimum.

Step 4. Find the possible points of inflection.

$18x + 14 = 0$

$x = -\frac{7}{9}$

$x$ B –0.8

Test the intervals.

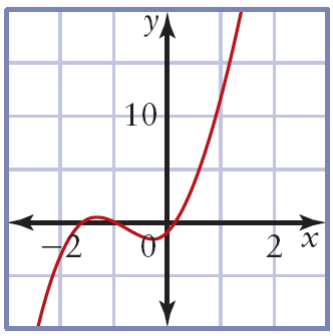
The intervals for $x = -\frac{7}{9}$ have already been tested.

Therefore (–0.8, –0.5) is a point of inflection.

Step 5. From Step 3, $k$ is increasing for $x < -1.3$ and $x > -0.3$ and decreasing for $-1.3 < x < -0.3$.

From Step 4, $k$ is concave down for $x < -0.8$ and concave up for $x > -0.8$.

Step 6. Sketch the graph.




**b)** Follow the six step plan.

$t(x) = 2x^3 - 12x^2 + 18x - 4$

$t'(x) = 6x^2 - 24x + 18$

$t''(x) = 12x - 24$

Step 1. Since $t$ is a polynomial function, the domain is $\{x \in \mathbb{R}\}$.

Step 2. $t(0) = -4$

The $y$-intercept is –4.

For $x$-intercepts, let $y = 0$.

$2x^3 - 12x^2 + 18x - 4 = 0$ Use factor theorem.

$t(1) = 4$

$t(-1) = -36$

$t(2) = 0$ $(x - 2)$ is a factor.

$(x - 2)(2x^2 - 8x + 2) = 0$

$x = 2,\ x = \dfrac{8 \pm \sqrt{48}}{4}$

$x = 2,\ x = 2 \pm \sqrt{3}$

The $x$-intercepts are 2 and $2 \pm \sqrt{3}$.

Step 3. Find the critical numbers.

$6x^2 - 24x + 18 = 0$

$6(x^2 - 4x + 3) = 0$

$(x - 3)(x - 1) = 0$

$x = 1,\ x = 3$

Use the second derivative test to classify the critical points.

$t''(1) = -12$

$t''(3) = 12$

Therefore, (1, 4) is a local maximum and (3, –4) is a local minimum.

Step 4. Find the possible points of inflection.

$12x - 24 = 0$

$x = 2$

Test the intervals. The intervals for $x = 2$ have been tested already.
Therefore, (2, 0) is a point of inflection.

Step 5. From Step 3, $t$ is increasing for $x < 1$ and $x > 3$ and decreasing for $1 < x < 3$.
From Step 4, $t$ is concave down for $x < 2$ and concave up for $x > 2$.

Step 6. Sketch the graph.

QuickTime™ and a
TIFF (Uncompressed) decompressor
are needed to see this picture.

($x \in [-4, 10]$, $y \in [-10, 10]$)

**c)** Follow the six step plan.

$f(x) = 2x^4 - 26x^2 + 72$

$f'(x) = 8x^3 - 52x$

$f''(x) = 24x^2 - 52$

Step 1. Since this is a polynomial function, the domain is $\{x \in \mathbb{R}\}$.

Step 2. $f(0) = 72$

The $y$-intercept is 72.

For $x$-intercepts, let $y = 0$.

$$2x^4 - 26x^2 + 72 = 0$$
$$2(x^4 - 13x^2 + 36) = 0$$
$$2(x^2 - 9)(x^2 - 4) = 0$$
$$2(x+3)(x-3)(x+2)(x-2) = 0$$

$x = 2, x = -2, x = 3, x = -3$

The $x$-intercepts are $\pm 2$ and $\pm 3$.

Step 3. Find the critical numbers.

$$8x^3 - 52x = 0$$
$$4x(2x^2 - 13) = 0$$

$x = 0,\ x = \sqrt{\frac{13}{2}},\ x = -\sqrt{\frac{13}{2}}$

$x = 9,\ x \approx 2.55,\ x \approx -2.55$

Use the second derivative test to classify the critical points.

$f''(-2.55) = 104$

$f''(2.55) = 104$

$f''(0) = -52$

Therefore, (0, 72) is a local maximum and (–2.55, –12.5) and (2.55, –12.5) are local minima.

Step 4. Find the possible points of inflection.

$24x^2 - 52 = 0$

$$x^2 = \frac{13}{6}$$

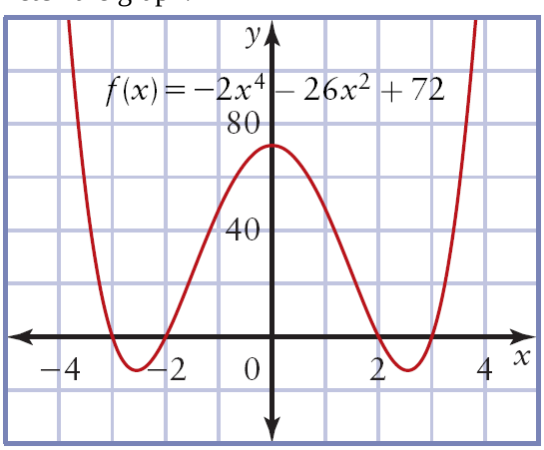
$$x = \sqrt{\frac{13}{6}},\ x = -\sqrt{\frac{13}{6}}$$

$x \approx 1.5,\ x \approx -1.5$

Test the intervals. The intervals for these values have already been tested.
Therefore, (–1.5, 25.2) and (1.5, 25.2) are points of inflection.

Step 5. From Step 3, $f$ is increasing for $-2.55 < x < 0$ and $x > 2.55$ and decreasing for $x < -2.55$ and $0 < x < 2.55$.
From ④, we know $f$ is concave down for $-1.5 < x < 1.5$ and concave up for $x < 1.5$ and $x > 1.5$.

Step 6. Sketch the graph.




**d)** Follow the six step plan.

$h(x) = 5x^3 - 3x^5$

$h'(x) = 15x^2 - 15x^4$

$h''(x) = 30x - 60x^3$

Step 1. Since this is a polynomial function, the domain is $\{x \in \mathbb{R}\}$.

Step 2. $h(0) = 0$

The $y$-intercept is 0.

For $x$-intercepts, let $y = 0$.

$$5x^3 - 3x^5 = 0$$
$$x^3(5 - 3x^2) = 0$$

$$x = 0, x = \sqrt{\frac{5}{3}}, x = -\sqrt{\frac{5}{3}}$$
$$x = 0, x \approx 1.3, x \approx -1.3$$

The $x$-intercepts are $\pm 1.3$ and 0.

Step 3. Find the critical numbers.

$$15x^2 - 15x^4 = 0$$
$$15x^2(1 - x^2) = 0$$

$$x = 0,\ x = 1,\ x = -1$$

Use the second derivative test to classify the critical points.

$h''(-1) = 30$

$h''(1) = -30$

$h''(0) = 0$ Check on either side of $x = 0$.

$h''(-0.5) = -7.5$

$h''(0.5) = 7.5$

Therefore, (–1, –2) is a local minimum and (1, 2) is a local maximum. (0, 0) is an inflection point.

Step 4. Find the possible points of inflection.

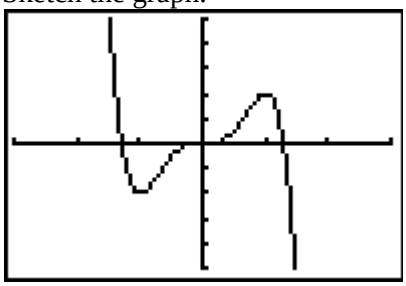
$$30x - 60x^3 = 0$$
$$30x(1 - 2x^2) = 0$$

$$x = 0,\ x = \sqrt{0.5},\ x = -\sqrt{0.5},$$
$$x = 0,\ x \approx 0.7,\ x \approx -0.7$$

Test the intervals. The intervals for these values have already been tested.
Therefore, (0, 0), (–0.7, –1.2) and (0.7, 1.2) are points of inflection.

Step 5. From Step 3, $h$ is increasing for $-1 < x < 0$ and $0 < x < 1$ and decreasing for $x < -1$ and $x > 1$.
From Step 4, $h$ is concave down for $-0.7 < x < 0$ and $x > 0.7$ and concave up for $x < -0.7$ and $0 < x < 1.7$.

Step 6. Sketch the graph.



**e)** Follow the six step plan.

$g(x) = 3x^4 + 2x^3 - 15x^2 + 12x - 2$

$g'(x) = 12x^3 + 6x^2 - 30x + 12$

$g''(x) = 36x^2 + 12x - 30$

Step 1. Since this is a polynomial function, the domain is $\{x \in \mathbb{R}\}$.

Step 2. $g(0) = -2$

The $y$-intercept is –2.

For $x$-intercepts, let $y = 0$.

$3x^4 + 2x^3 - 15x^2 + 12x - 2 = 0$

$f(1) = 0$ $(x - 1)$ is a factor.

$f(-1) = -28$

$f(0.5) = 0.6875$

$f(0.25) \approx 0.1055$

$f(0.23) \approx -0.0007$

$3x^4 + 2x^3 - 15x^2 + 12x - 2$

$= (x+1)(3x^3 - x^2 - 14x + 26) + -28$

The second factor is not easily factorable.

The $x$-intercept is 1.

Step 3. Find the critical numbers.

$$12x^3 + 6x^2 - 30x + 12 = 0$$

$$6(2x^3 + x^2 - 5x + 2) = 0$$ Use the factor theorem.

$$f(1) = 0$$ $(x - 1)$ is a factor.

$$6(x-1)(2x^2 + 3x - 2) = 0$$

$$6(x-1)(2x-1)(x+2) = 0$$

$$x = 1,\ x = -2,\ x = \frac{1}{2}$$

Use the second derivative test to classify the critical points.

$$g''(1) = 18$$

$$g''(-2) = 90$$

$$g''(0.5) = -15$$

Therefore, (–2, –54) and (1, 0) are local minimum points. (0.5, 0.7) is a local maximum.

Step 4. Find the possible points of inflection.

$$36x^2 + 12x - 30 = 0$$

$$6(6x^2 + 2x - 5) = 0$$

$$x = \frac{-2 \pm \sqrt{124}}{12}$$

$$= \frac{-1 \pm \sqrt{31}}{6}$$

$$x \approx -1.1,\ x \approx 0.8$$

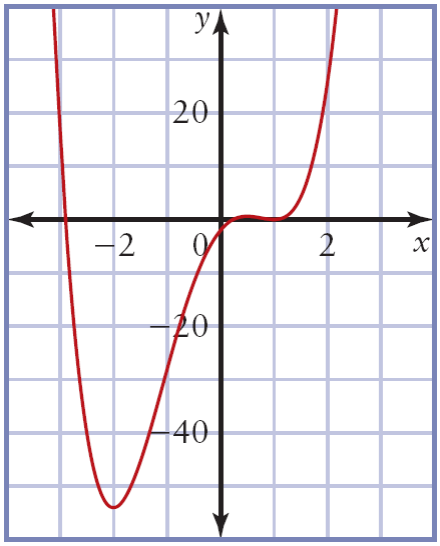
Test the intervals. The intervals for these values have already been tested.

Therefore, (–1.1, –32) and (0.8, 0.3) are points of inflection.

Step 5. From Step 3, $g$ is increasing for $-2 < x < 0.5$ and $x > 1$ and decreasing for $x < -2$ and $0.5 < x < 1$.

From Step 4, $g$ is concave down for $-1.1 < x < 0.8$ and concave up for $x < -1.1$ and $x > 0.8$.

Step 6. Sketch the graph.



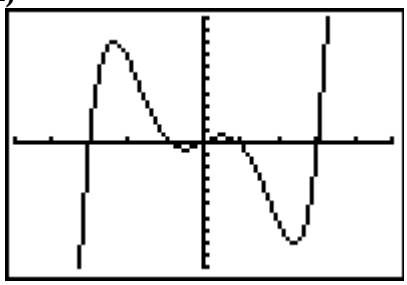

**Chapter 3 Section 5 Question 10 Page 193**

This is false. For example, constant functions have no turning points. Even polynomial functions of higher degrees may not have turning points. The function $f(x) = x^3$ has a point of inflection at the origin, but no turning points.

**Chapter 3 Section 5 Question 11 Page 193**

**a)** $f(x) = 2x^5 - 20x^3 + 15x$

**i)**



**(** $x \in [-5, 5]$, $y \in [-100, 100]$, $\text{Yscl} = 10$ **)**

**ii)** Points of inflection appear to be at the origin and also when $x = -1.5$ or $x = 1.5$
Thus the function appears to be concave down for $x < -1.5$ and $0 < x < 1.5$. The curve appears to be concave up for $-1.5 < x < 0$ and $x > 1.5$.

**iii)** $f'(x) = 10x^4 - 60x^2 + 15$

$f''(x) = 40x^3 - 120x$

For inflection points, let $f''(x) = 0$

$40x^3 - 120x = 0$

$40x(x^2 - 3) = 0$

$x = 0,\ x = \sqrt{3},\ x = -\sqrt{3}$

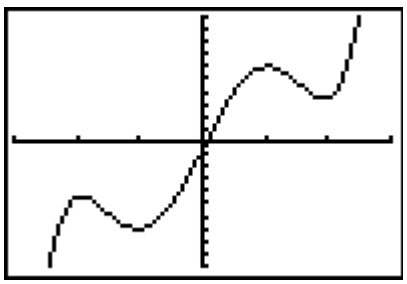
Therefore, the actual inflection points are (0, 0), $\left(-\sqrt{3},\ 46.77\right)$ and $\left(\sqrt{3},\ -46.77\right)$.

Also, the function is concave down for $x < -\sqrt{3}$ and $0 < x < \sqrt{3}$ and concave up for $-\sqrt{3} < x < 0$ and $x > \sqrt{3}$.

Thus the function is concave down for $x < -1.5$ and $0 < x < 1.5$. The curve is concave up for $1.5 < x < 0$ and $x > 1.5$.

**b)** $g(x) = x^5 - 8x^3 + 20x - 1$

**i)**



$(x \in [-3,\ 3],\ y \in [-29,\ 20],\ \text{Yscl} = 2)$

**ii)** Points of inflection appear to be at the origin and also when $x = -1.5$ or $x = 1.5$.
Thus the function appears to be concave down for $x < -1.5$ and $0 < x < 1.5$. The curve appears to be concave up for $-1.5 < x < 0$ and $x > 1.5$.

**iii)** $f'(x) = 5x^4 - 24x^2 + 20$

$f''(x) = 20x^3 - 48x$

For inflection points, let $f''(x) = 0$.

$20x^3 - 48x = 0$

$4x(5x^2 - 12) = 0$

$x = 0,\ x = \sqrt{2.4},\ x = -\sqrt{2.4}$

Therefore, the actual inflection points are (0, 0), $\left(-\sqrt{2.4},\ 7.66\right)$ and $\left(\sqrt{2.4},\ 2.94\right)$.

Also, the function is concave down for $x < -\sqrt{2.4}$ and $0 < x < \sqrt{2.4}$ and concave up for $-\sqrt{2.4} < x < 0$ and $x > \sqrt{2.4}$.

**Chapter 3 Section 5 Question 12 Page 193**

Many different functions are possible.
Since $x = 2$ is a critical value, $(x - 2)$ must be a factor of the first derivative.
Since there is a point of inflection when $x = 0$, $x$ must be a factor of $f''(x)$.

Try $f'(x) = (x - 2)$. This does not work since $f''(x) = 1$ (which does not have $x$ as a factor).

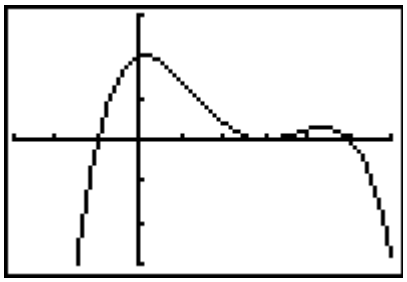
Try $f'(x) = (x - 2)(x + 2)$
$= x^2 - 4$
This works since $f''(x) = 2x$ (which has $x$ as a factor).

A possible function is $f(x) = \frac{1}{3}x^3 - 4x + k$ which has the necessary first and second derivatives.
(Note that $k$ can be any constant value.)

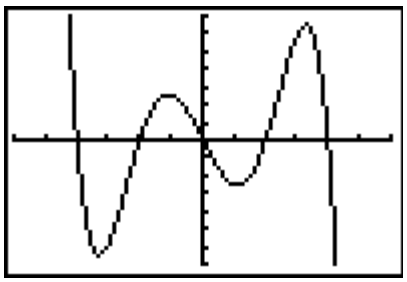
**Chapter 3 Section 5 Question 13 Page 193**

**a)** The function must be increasing to $x$-intercept –1, continuing to increase through the $y$-intercept of 2 until it reaches a local maximum at $x = 2$. It then decreases to the point (3, 0) which is a local minimum. Then it increases to a local maximum at $x = 4$. Finally it descends through the $x$-intercept 5 and continues descending thereafter.

A possible equation is $f(x) = -\frac{2}{45}(x + 1)(x - 3)^2(x - 5)$.



**b)** The 5 intercepts are –4, –2, 2, 4, and 0. The missing ones are added because the function is odd and must be symmetric about the origin. The function starts in the second quadrant, descends through the $x$-intercept –4, increases through the (–1, 5) and then descends to the origin. The right half of the graph is a reflection in the origin of this path.
The function is of the form $g(x) = kx(x^2 - 4)(x^2 - 16)$.
The local point (–1, 5) determines the scale factor $k$.
A possible equation is $g(x) = -\frac{1}{9}x(x^2 - 4)(x^2 - 16)$.

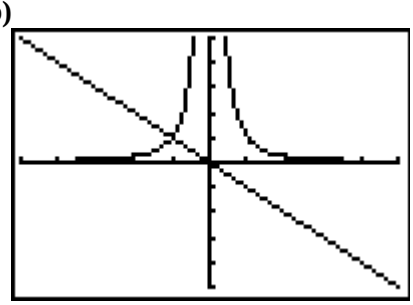


$(x \in [-6, 6],\ y \in [-14, 14],\ \text{Yscl} = 2)$

**Chapter 3 Section 5 Question 14 Page 193**

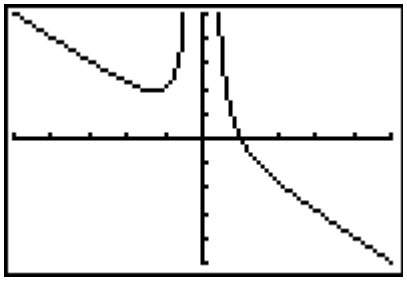
**a)** Two possible functions are $g(x) = -x$ and $h(x) = \frac{1}{x^2}$.

**b)**



**c)** The graph will look more like $h(x)$ in the centre of the graph but more like $g(x)$ farther from the origin. The sum will have a vertical asymptote since $h(x)$ has one. The horizontal asymptote will be 'bent' by the influence of $g(x)$. In fact, the sum curve will get closer and closer to $g(x)$ as one gets farther from the origin ($g(x)$ will become an asymptote for the sum curve).

**d)**



**e)** $f(x) = -x + x^{-2}$

$$f'(x) = -1 - 2x^{-3}$$

$$= -1 - \frac{2}{x^3}$$

For a critical point,

$$-1-\frac{2}{x^3}=0$$
$$x^3-2=0$$
$$x=\sqrt[3]{2}$$

The turning point is $\left(-\sqrt[3]{2}, -\sqrt[3]{2}-\frac{1}{\sqrt[3]{4}}\right)$

**f)** $f''(x)=6x^{\sqrt{-4}}$

$$=\frac{6}{x^4}$$

There are no inflection points ( $x=0$ is not in the domain of *f*).

The vertical asymptote may divide areas of different concavity.
Test the intervals.

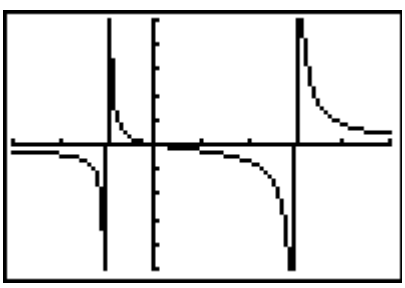
$f''(-1)=6$

$f''(1)=6$

The curve is concave up for $x<0$ and $x>0$.

**Chapter 3 Section 5 Question 15 Page 193**

**a)**



**b)** Yes. The function can be stretched vertically by any positive factor.

One possible equation is $y=\frac{x}{(x+1)(x-3)}$.

**c)** It is reasonable to assume that there could be a minimum in the concave up interval or a maximum in the concave down interval. However, finding such an equation is not simple. It likely involves function types not studied in this course.

**Chapter 3 Section 5 Question 16 Page 193**

**a)** $\lim_{x\to\infty} g(x)=0$ and $\lim_{x\to-\infty} g(x)=0$

This means that the *y*-axis is a horizontal asymptote for this function. This result can be verified by testing large (positive and negative) values of *x*.

$$g(100) = \frac{1}{(100)^2 - 1}$$

B 0.0001

$$g(\breve{G}100) = \frac{1}{(\breve{G}100)^2 - 1}$$

$\sqcap$ B 0.0001

**b)** The vertical asymptotes are $x = -1$ and $x = 1$.

**c)** Check $x$ = 1.

$$g(1.01) = \frac{1}{(1.01)^2 - 1}$$

B 50

$$g(1.001) = \frac{1}{(1.001)^2 - 1}$$

B 500

$$g(0.99) = \frac{1}{(0.99)^2 - 1}$$

B –50

$$g(0.999) = \frac{1}{(0.999)^2 - 1}$$

$\sqcap$ B –500

Therefore, $\lim\limits_{x \to 1^-} g(x) = -\infty$ and $\lim\limits_{x \to 1^+} g(x) = +\infty$.

**d)** The function is an even function. It is symmetric about the $y$-axis. This allows us to determine the limits for the other asymptote directly.

$\lim\limits_{x \to -1^-} g(x) = -\infty$ and $\lim\limits_{x \to -1^+} g(x) = +\infty$.

**e)** $g(x) = (x^2 - 1)^{-1}$

$$\begin{aligned} g'(x) &= -2x(x^2 - 1)^{-2} \\ &= \frac{-2x}{(x^2 - 1)} \end{aligned}$$

$$\begin{aligned} g''(x) &= -2(x^2 - 1)^{-2} + (-2x)(-2)(x^2 - 1)^{-3}(2x) \\ &= (x^2 - 1)^{-3}(-2x^2 + 2 + 8x^2) \\ &= \frac{6x^2 + 2}{(x^2 - 1)^3} \end{aligned}$$

For critical points,

$$\frac{-2x}{(x^2-1)}=0$$
$$x=0$$

Check concavity with the second derivative.

$$g''(0)=\frac{6(0)^2+2}{((0)^2-1)^3}$$
$$=-2$$

The point (0, –1) is a local maximum.

**Chapter 3 Section 5 Question 17 Page 194**

**a)** Let $x = 0$.

$f(0)=-1$.

The $y$-intercept is –1.

Let $y = 0$.

$$\frac{x-1}{x+1}=0$$
$$x-1=0$$
$$x=1$$

The $x$-intercept is 1.

**b)** The vertical asymptote is found by letting the denominator equal zero. The asymptote is $x=-1$.

A horizontal asymptote can be found by examining limits at infinity.

$$\lim_{x\to\infty} f(x)=\lim_{x\to\infty}\frac{x-1}{x+1}$$ Limit is indeterminate: $\frac{\infty}{\infty}$

$$=\lim_{x\to\infty}\frac{1-\frac{1}{x}}{1+\frac{1}{x}}$$ Divide by $x$ in the numerator and denominator.

$$=1$$

$$\lim_{x\to-\infty} f(x)=\lim_{x\to-\infty}\frac{x-1}{x+1}$$ Limit is indeterminate: $\frac{-\infty}{-\infty}$

$$=\lim_{x\to-\infty}\frac{1-\frac{1}{x}}{1+\frac{1}{x}}$$ Divide by $x$ in the numerator and denominator.

$$=1$$

There is a horizontal asymptote at $y = 1$.

**c)** $f(x) = (x-1)(x+1)^{-1}$

$$f'(x) = 1(x+1)^{-1} + (x-1)(-1)(x+1)^{-2}(1)$$
$$= (x+1)^{-2}(x+1-x+1)$$
$$= \frac{2}{(x+1)^2}$$

There are no critical values.

However, you need to test the intervals on either side of the asymptote.

$f'(-2) = 2$

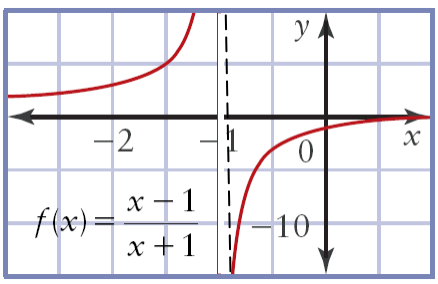
$f'(1) = \frac{1}{2}$

The function is increasing for $x < -1$ and $x > -1$.

**d)** Since the function is increasing as it approaches the asymptote from the left, the $\lim_{x \to -1^-} f(x) = \infty$.

Since the function is decreasing as it approaches the asymptote from the right, the $\lim_{x \to -1^+} f(x) = -\infty$.

**e)**




**Chapter 3 Section 5** **Question 18 Page 194**

**a)** $g(x) = x^3 - 27x$

$g'(x) = 3x^2 - 27$

$g''(x) = 6x$

**i)** This is a polynomial function whose 'end' behaviour is determined by the sign of the coefficient of the term of highest degree. In this case, the coefficient is +1.

Therefore, $\lim_{x \to \infty} g(x) = \infty$ and $\lim_{x \to -\infty} g(x) = -\infty$.

**ii)** Examine the following.

$$g(-x) = (-x)^3 - 27(-x)$$
$$= -x^3 + 27x$$
$$= -g(x)$$

The function is an odd function and is symmetric about the origin.

**iii)** Find critical values.

$$3x^2 - 27 = 0$$
$$x^2 = 9$$
$$x = 3, x = -3$$

Test using the second derivative test.

$$g''(-3) = 6(-3)$$
$$= -18$$
$$g''(3) = 6(3)$$
$$= 18$$

Point (–3, 54) is a local maximum and (3, –54) is a local minimum.

**iv)** Find possible points of inflection.

$$6x = 0$$
$$x = 0$$

| | $x<0$ | $x=0$ | $x>0$ |
|---|---|---|---|
| $g''(x)$ | Negative | 0 | Positive |
| $g(x)$ | Concave down | Point of inflection | Concave up |

There is a point of inflection at (0, 0).

**b)** $y = f(x)$

$$= x^4 - 8x^2 + 16$$
$$f'(x) = 4x^3 - 16x$$
$$f''(x) = 12x^2 - 16$$

**i)** This is a polynomial function whose 'end' behaviour is determined by the sign of the coefficient of the term of highest degree. In this case, the coefficient is +1.

Therefore, $\lim_{x \to \infty} f(x) = \infty$ and $\lim_{x \to -\infty} f(x) = -\infty$.

**ii)** Examine the following.

$$f(-x) = (-x)^4 - 8(-x)^2 + 16$$
$$= x^4 - 8x^2 + 16$$
$$= f(x)$$

The function is an even function as so it is symmetric about the *y*-axis.

**iii)** Find critical values.

$$4x^3 - 16x = 0$$
$$4x(x^2 - 4) = 0$$
$$x = 0,\ x = 2,\ x = -2$$

Test using the second derivative test.

$$f''(-2) = 12(-2)^2 - 16$$
$$= 32$$
$$f''(2) = 12(2)^2 - 16$$
$$= 32$$
$$f''(0) = 12(0)^2 - 16$$
$$= -16$$

Point (0, 16) is a local maximum and points (–2, 0) and (2, 0) are local minimums.

**iv)** Find possible points of inflection.

$$12x^2 - 16 = 0$$
$$x^2 = \frac{4}{3}$$

$$x = \frac{2}{\sqrt{3}},\ x = -\frac{2}{\sqrt{3}}$$
$$x \approx 1.15,\ x \approx -1.15$$

| | $x < \sqrt{\frac{4}{3}}$ | $x < -\sqrt{\frac{4}{3}}$ | $-\sqrt{\frac{4}{3}} < x < +\sqrt{\frac{4}{3}}$ | $x = +\sqrt{\frac{4}{3}}$ | $x > +\sqrt{\frac{4}{3}}$ |
|---|---|---|---|---|---|
| $y''$ | Positive | 0 | Negative | 0 | Positive |
| $y$ | Concave up | Point of inflection | Concave down | Point of inflection | Concave up |

There are points of inflection at $\left(-\frac{2}{\sqrt{3}}, \frac{64}{9}\right)$ and $\left(\frac{2}{\sqrt{3}}, \frac{64}{9}\right)$.

**c)** $k(x) = \frac{1}{1-x^2}$
$$= (1-x^2)^{-1}$$
$$k'(x) = -(1-x^2)^{-2}(-2x)$$
$$= \frac{2x}{(1-x^2)^2}$$
$$k''(x) = 2(1-x^2)^{-2} + (2x)(-2)(1-x^2)^{-3}(-2x)$$
$$= (1-x^2)^{-3}(2 - 2x^2 + 8x^2)$$
$$= \frac{6x^2+2}{(1-x^2)^3}$$

**i)** Take limits.

$$\lim_{x\to\infty} k(x) = \lim_{x\to\infty} \frac{1}{1-x^2} = 0$$ 1 divided by a large positive number.

$$\lim_{x\to-\infty} k(x) = \lim_{x\to-\infty} \frac{1}{1-x^2} = 0$$ 1 divided by a large positive number.

**ii)** Examine $k(-x) = \dfrac{1}{1-(-x)^2} = \dfrac{1}{1-x^2} = k(x)$.

The function is an even function as so it is symmetric about the *y*-axis.

**iii)** Find critical values.

$$\frac{2x}{(1-x^2)^2} = 0$$

$$x = 0$$

Test using the second derivative test.

$$k''(0) = \frac{6(0)^2+2}{(1-(0)^2)^3}$$

$$= 2$$

Point (0, 1) is a local minimum.

**iv)** Find possible points of inflection.

$$\frac{6x^2+2}{(1-x^2)^3} = 0$$

$$6x^2+2 = 0$$

There are no real roots to this equation. There are no points of inflection.
Concavity can only change at the vertical asymptotes ( $x = \neq 1$ ) that are present in this case.

| | $x<-1$ | $x=-1$ | $-1<x<1$ | $x=1$ | $x>1$ |
|---|---|---|---|---|---|
| $k''(x)$ | Negative | Undefined | Positive | Undefined | Negative |
| $k(x)$ | Concave down | Vertical asymptote | Concave up | Vertical asymptote | Concave down |

**d)** $f(x)=\dfrac{x}{x^2+1}$

$$=x(x^2+1)^{-1}$$

$$f'(x)=1(x^2+1)^{-1}+x(\check{G}1)(x^2+1)^{-2}(2x)$$

$$=(x^2+1)^{-2}(x^2+1-2x^2)$$

$$=\frac{-x^2+1}{(x^2+1)^2}$$

$$f''(x)=-2x(x^2+1)^{-2}+(-x^2+1)\left(-2\right)(x^2+1)^{-3}(2x)$$

$$=(x^2+1)^{-3}(-2x^3-2x+4x^3-4x)$$

$$=\frac{-6x+2x^3}{(x^2+1)^3}$$

**i)** Take limits.

$$\lim_{x\to\infty} f(x)=\lim_{x\to\infty}\frac{x}{x^2+1}$$

$$=\lim_{x\to\infty}\frac{\frac{1}{x}}{x+\frac{1}{x}} \qquad \text{Divide by } x^2/x^2.$$

$$=0$$

$$\lim_{x\to-\infty} f(x)=\lim_{x\to-\infty}\frac{x}{x^2+1}$$

$$=\lim_{x\to-\infty}\frac{\frac{1}{x}}{x+\frac{1}{x}} \qquad \text{Divide by } x^2/x^2.$$

$$=0$$

**ii)** Examine the following.

$$f(-x)=\frac{(-x)}{(-x)^2+1}$$

$$=\frac{-x}{x^2+1}$$

$$=-f(x)$$

The function is an odd function so it is symmetric about the origin.

**iii)** Find critical values.

$$\frac{-x^2+1}{(x^2+1)^2}=0$$

$$x^2=1$$

$$x=1,\ x=\check{G}1$$

Test using the second derivative test.

$$f''(-1) = \frac{-6(-1) + 2(-1)^3}{((-1)^2 + 1)^3}$$

$$= \frac{4}{8}$$

$$f''(1)^2 = \frac{-6(1) + 2(1)^3}{((1)^2 + 1)^3}$$

$$= \frac{-4}{8}$$

Point $\left(-1, -\frac{1}{2}\right)$ is a local minimum and $\left(1, \frac{1}{2}\right)$ is a local maximum.

**iv)** Find possible points of inflection.

$$\frac{-6x + 2x^3}{(x^2 + 1)^3} = 0$$

$$-6x + 2x^3 = 0$$

$$2x(x^2 - 3) = 0$$

$$x = 0,\ x = \sqrt{3},\ x = \check{G}\sqrt{3}$$

There are three possible points of inflection.

<<There are 6 instances of a square root sign where the plus/minus sign is misplaced in front of it. $\pm\sqrt{3}$ is correct placement.>><<pick up corrected artwork from answers>>

| | $x < -\sqrt{3}$ | $x = -\sqrt{3}$ | $-\sqrt{3} < x < 0$ | $x = 0$ |
|---|---|---|---|---|
| $f(x)''$ | Negative | 0 | Positive | 0 |
| ***f*(*x*)** | Concave down | Point of inflection | Concave up | Point of inflection |

| | $0 < x < +\sqrt{3}$ | $x = +\sqrt{3}$ | $x > +\sqrt{3}$ |
|---|---|---|---|
| $f(x)''$ | Negative | 0 | Positive |
| ***f*(*x*)** | Concave down | Point of inflection | Concave up |

The points of inflection are $\left(-\sqrt{3}, -\frac{\sqrt{3}}{4}\right)$, $\left(\sqrt{3}, \frac{\sqrt{3}}{4}\right)$, and $(0, 0)$.

**e)** $h(x)=\dfrac{x-4}{x^2}=x^{-2}(x-4)$

$$\begin{aligned}h'(x)&=x^{-2}(1)+(-2)x^{-3}(x-4)\\&=x^{-3}(x-2x+8)\\&=\frac{8-x}{x^3}\end{aligned}$$

$$\begin{aligned}h'(x)&=x^{-3}(-1)-3x^{-4}(8-x)\\&=x^{-4}(-x+3x-24)\\&=\frac{2x-24}{x^4}\end{aligned}$$

**i)** Take limits.

$$\begin{aligned}\lim_{x\to\infty}h(x)&=\lim_{x\to\infty}\frac{x-4}{x^2}\\&=\lim_{x\to\infty}\frac{\frac{1}{x}-\frac{4}{x^2}}{1}\qquad\text{Divide by } x^2/x^2.\\&=0\end{aligned}$$

$$\begin{aligned}\lim_{x\to-\infty}h(x)&=\lim_{x\to-\infty}\frac{x-4}{x^2}\\&=\lim_{x\to-\infty}\frac{\frac{1}{x}-\frac{4}{x^2}}{1}\qquad\text{Divide by } x^2/x^2.\\&=0\end{aligned}$$

**ii)** Examine the following.

$$\begin{aligned}h(-x)&=\frac{(-x)-4}{(-x)^2}\\&=\frac{-x-4}{x^2}\end{aligned}$$

The function is neither odd nor even. There is no resulting symmetry.

**iii)** Find critical values.

$$\begin{aligned}\frac{8-x}{x^3}&=0\\8-x&=0\\x&=8\end{aligned}$$

Test using the second derivative test.

$$\begin{aligned}h''(8)&=\frac{2(8)-24}{(8)^4}\\&=-0.002\end{aligned}$$

Point $\left(8, \frac{1}{16}\right)$ is a maximum

**iv)** Find possible points of inflection.

$$\frac{2x-24}{x^4}=0$$
$$2x-24=0$$
$$x=12$$

There is 1 possible point of inflection.

| | $x<0$ | $x=0$ | $0<x<12$ | $x=12$ | $x>12$ |
|---|---|---|---|---|---|
| $h''(x)$ | Negative | Undefined | Negative | 0 | Positive |
| $h(x)$ | Concave Down | Vertical Asymptote | Concave Down | Point of Inflection | Concave Up |
| | | | | | |

The point of inflection is $\left(12, \frac{1}{18}\right)$.

**Chapter 3 Section 5 Question 19 Page 194**

**a)** There is no $x$-intercept since the fraction can never equal zero. The $y$-intercept is 1 since

$$g(0)=\frac{1}{0^2+1}$$
$$=1$$

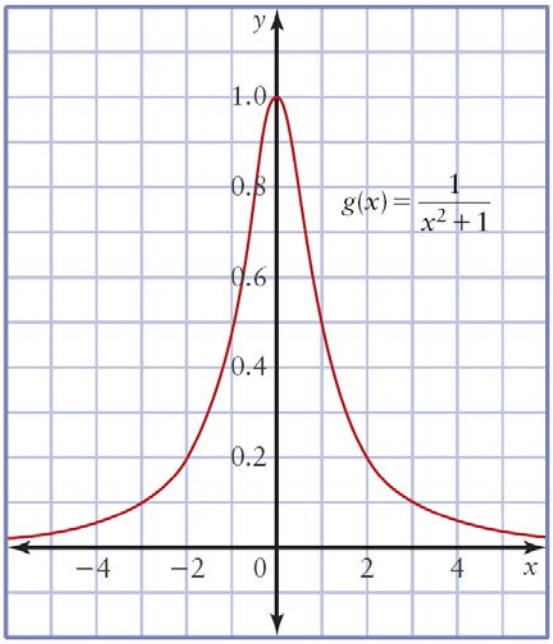
**b)** The maximum value for $g(x)$ is 1. The function $g(x)$ is a fraction whose numerator is fixed as 1. For a maximum value, the denominator must be as small as possible. Since $x^2$ is always greater than or equal to zero, the smallest the denominator can be is 0 + 1 = 1. Therefore, the maximum value for the fraction is 1.

**c)** For large (positive or negative) values of $x$, $g(x)$ becomes 1 over a very large number, which gets closer and closer to the value zero. The horizontal asymptote is the $y$-axis or $y=0$.

**d)** No. The function has a positive $y$-intercept that is its maximum. As $x$ gets large, the function decreases and approaches the $x$-axis from above. It is very unlikely that there would be any extrema in this situation. (However, it might be possible to imagine a damped sinusoidal curve that could satisfy the conditions of parts a) to c) of this question, but still have multiple local maxima and minima as the function approaches the $x$-axis.)
Clearly, using the tools of calculus (e.g., derivatives) would verify this conjecture.

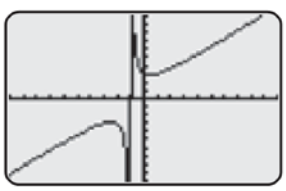
**e)** Two. The function has one maximum where the function must be concave down. At the limit, the function must be concave up (or it will intersect the axis). There must be one point of inflection between the maximum and the end of the $x$-axis in either direction.

**f)**



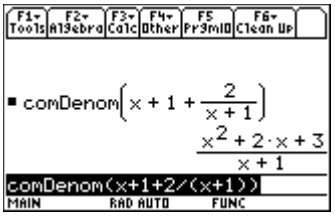

**Chapter 3 Section 5 Question 20 Page 194**

**a)**



**b)**
$$\begin{aligned}\frac{x^2+2x+3}{x+1} &= \frac{x^2+2x+1+2}{x+1}\\ &= \frac{x^2+2x+1}{x+1}+\frac{2}{x+1}\\ &= \frac{(x+1)^2}{x+1}+\frac{2}{x+1}\\ &= x+1+\frac{2}{x+1}\end{aligned}$$

Another method is to use a CAS.



**c)** The two functions are $s(x) = x + 1$ and $t(x) = \dfrac{2}{x+1}$.

The function *t* has a vertical asymptote; therefore *f* will also have the same vertical asymptote. Function *t* approaches the value zero as *x* increases; therefore *f* will approach the curve *s* as *x* increases. That is, *s* will be a slant asymptote for *f*.

**Chapter 3 Section 5 Question 21 Page 194**

C is the correct answer.

$$y = (x+2)^5(x^2-1)^4$$
$$\frac{dy}{dx} = (x+2)^4(x^2-1)^3(13x^2+16x-5)$$

When $\dfrac{dy}{dx} = 0$, $x = -2$ or $\pm 1$, or two other real roots from the quadratic equation $13x^2 + 16x - 5 = 0$.

With the exception of $-2$, $\dfrac{dy}{dx}$ will have a sign change when passing through each of these roots. Hence, there are four local maximum or minimum points.

**Chapter 3 Section 5 Question 22 Page 194**

D is the correct answer.

The functions $y = \dfrac{1}{x^2}$, $y = \dfrac{1}{x}$, $y = \dfrac{x^2}{x}$, and $y = \dfrac{1}{\sqrt{x}}$ are discontinuous at $x = 0$.
Hence, their derivatives are also discontinuous at $x = 0$.
However, $x = 0$ is not in the domains of these functions.
The function $y = \sqrt{x}$ has derivative $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{x}}$, which is discontinuous at $x = 0$.
In this case, $x = 0$ is in the domain of the function.

**Chapter 3 Section 6 Optimization Problems**

**Chapter 3 Section 6 Question 1 Page 201**

For a maximum, find the first derivative and let it equal zero to find the critical points.

$$h(t) = -4.9t^2 + 19.6t + 2$$
$$h'(t) = -9.8t + 19.6$$
$$0 = -9.8t + 19.6$$
$$t = 2$$

Since this function is a parabola opening down, this critical point is a maximum.

$$h(2) = -4.9(2)^2 + 19.6(2) + 2$$
$$= 21.6$$

The maximum height of the ball is 21.6 m.

**Chapter 3 Section 6 Question 2 Page 201**

Let the integers be represented by $x$ and $20 - x$ and $P$ be their product (the quantity to be maximized).

$$P(x) = x(20 - x)$$
$$= 20x - x^2$$
$$P'(x) = 20 - 2x$$

Find the critical points.

$$20 - 2x = 0$$
$$x = 10$$

Since this function is a parabola opening down, this critical point is a maximum.
The two integers are 10 and (20 – 10) = 10.

**Chapter 3 Section 6 Question 3 Page 201**

Find the critical points.

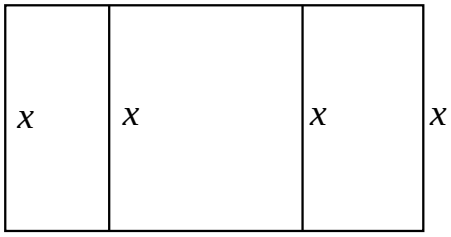
$$N(t) = -0.05t^2 + 3t + 5$$
$$N'(t) = -0.1t + 3$$
$$0 = -0.1t + 3$$
$$t = 30$$

Since this function is a parabola opening down, this critical point is a maximum.
It takes 30 years of experience to reach maximum productivity.

**Chapter 3 Section 6** **Question 4 Page 201**

**a)** Let the length of each small pen be $x$ metres. This uses $4x$ of fencing.
$1200 - 4x$ of fencing remains for the two long sides.
Each long side has length:
$\frac{1200-4x}{2} = 600 - 2x$.

Let $A$ be the area enclosed.




$$\begin{aligned} A(x) &= x(600-2x) \\ &= 600x - 2x^2 \end{aligned}$$

Find the critical points.

$$\begin{aligned} A'(x) &= 600 - 4x \\ 0 &= 600 - 4x \\ x &= 150 \end{aligned}$$

Since this function is a parabola opening down, this critical point is a maximum.

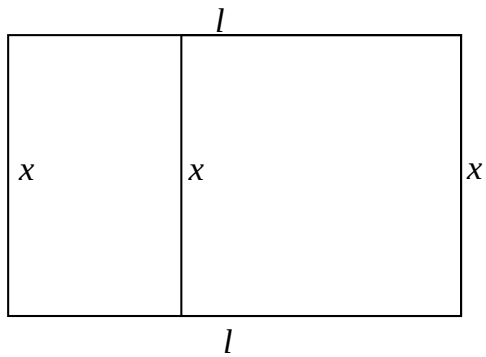
$$\begin{aligned} A(150) &= 600(150) - 2(150)^2 \\ &= 45\ 000 \end{aligned}$$

The maximum area that can be enclosed is 45 000 $m^2$.

**b)** This condition would limit the function to the interval: $180 < x < 210$. (The right hand limit occurs because the long sides of the big pen must also be at least 180 m long and that would leave only 840 m for the four shorter sides.)
Consequently, there would be no critical point in the interval. A maximum value would have to occur at an endpoint. Since $A(180) = 43\ 200$ and $A(210) = 37\ 800$, the maximum possible area would be 43 200 $m^2$ in this case.

**Chapter 3 Section 6** **Question 5 Page 201**

Let the width of the square be $x$ metres.
Let $l$ be the length of the combined pen.




$$3x + 2l = 60$$
$$l = \frac{60 - 3x}{2}$$

Let $A$ be the overall area.

$$A(x) = x\left(\frac{60 - 3x}{2}\right)$$
$$= 30x - \frac{3}{2}x^2$$

Find the critical points.

$$A'(x) = 30 - 3x$$
$$0 = 30 - 3x$$
$$x = 10$$

Since this function is a parabola opening down, this critical point is a maximum.

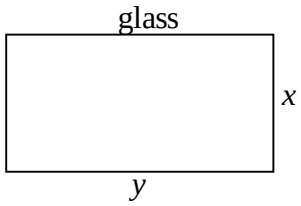
$$A(10) = 30(10) - \frac{3}{2}(10)^2$$
$$= 150$$

$$l = \frac{60 - 3(10)}{2}$$
$$= 15$$

The maximum area possible is 150 $m^2$ and the dimensions that give this area are 10 m by 15 m.

**Chapter 3 Section 6** **Question 6 Page 201**

**a)** Let the dimensions of the showroom be $x$ and $y$.




$$xy = 500$$
$$y = \frac{500}{x}$$

Let the total cost be represented by $C$.

$$\begin{aligned} C(x) &= 600\left(\frac{500}{x}\right) + 1200\left(\frac{500}{x} + 2x\right) \\ &= 2400x + \frac{900\ 000}{x} \\ &= 2400x + 900\ 000x^{-1} \end{aligned}$$

Find the critical points.

$$\begin{aligned} C'(x) &= 2400 - 900\ 000x^{-2} \\ 0 &= 2400 - 900\ 000x^{-2} \\ 0 &= 2400x^2 - 900\ 000 \\ x^2 &= 375 \\ x &= \pm\sqrt{375} \\ x &= \pm 5\sqrt{15} \end{aligned}$$

Considering the context, $0 < x < 500$. Check the total cost at $x = 5\sqrt{15}$ and the endpoints.

$$\begin{aligned} C\left(5\sqrt{15}\right) &= 2400\left(5\sqrt{15}\right) + \frac{900\ 000}{\left(5\sqrt{15}\right)} \\ &= 92\ 952 \\ C(1) &= 2400(1) + \frac{900\ 000}{(1)} \\ &= 902\ 400 \\ C(500) &= 2400(500) + \frac{900\ 000}{(500)} \\ &= 1\ 200\ 000 \end{aligned}$$

The minimum cost occurs when the dimensions are $5\sqrt{15}$ and $\frac{500}{5\sqrt{15}} = \frac{20\sqrt{15}}{3}$.

The showroom should be about 19.4 m by 25.8 m.

**b)** Roofing will have a fixed cost since its area (500 m$^2$) does not change as different showroom rectangles are considered.

**Chapter 3 Section 6** **Question 7 Page 201**

**a)** Let the perimeter be $P$. Then $P = 2h + 2r + \pi r$ and so $h = \frac{1}{2}(P - 2r - \pi r)$.

$$\begin{aligned}
A &= \frac{\pi r^2}{2} + 2rh \\
&= \frac{\pi r^2}{2} + 2r\left(\frac{P - 2r - \pi r}{2}\right) \\
&= \frac{\pi r^2}{2} + Pr - 2r^2 - \pi r^2 \\
A' &= \pi r + P - 4r - 2\pi r \\
A' &= P - 4r - \pi r \\
P - 4r - \pi r &= 0 \\
r &= \frac{P}{4+\pi}
\end{aligned}$$

$$\begin{aligned}
h &= \frac{1}{2}(P - 2r - \pi r) \\
&= \frac{1}{2}\left[P - 2\left(\frac{P}{4+\pi}\right) - \pi\left(\frac{P}{4+\pi}\right)\right] \\
&= \frac{P}{2}\left(1 - \frac{2+\pi}{4+\pi}\right) \\
&= \frac{P}{2}\left(\frac{4+\pi}{4+\pi} - \frac{2+\pi}{4+\pi}\right) \\
&= \frac{P}{2}\left(\frac{2}{4+\pi}\right) \\
&= \frac{P}{4+\pi}
\end{aligned}$$

So the ratio is:

$$\begin{aligned}
\frac{h}{r} &= \frac{\frac{P}{4+\pi}}{\frac{P}{4+\pi}} \\
&= 1
\end{aligned}$$

**b)** Let $A$ be the area and assume the cost for the straight edges is \$$x$/m.

$$A = \frac{\pi r^2}{2} + 2rh$$

$$h = \frac{1}{2r}\left(A - \frac{\pi r^2}{2}\right)$$

$$= \frac{A}{2r} - \frac{\pi r}{4}$$

$$C(x) = x(2h + 2r) + 3x(\pi r)$$

$$= 2x\left[\frac{A}{2r} - \frac{\pi r}{4}\right] + 2xr + 3x\pi r$$

$$= x\left(\frac{A}{r} - \frac{\pi r}{2} + 2r + 3\pi r\right)$$

$$C'(x) = x\left(\frac{-A}{r^2} - \frac{\pi}{2} + 2 + 3\pi\right)$$

$$0 = x\left(\frac{-A}{r^2} + \frac{5\pi}{2} + 2\right)$$

$$\frac{A}{r^2} = \frac{5\pi}{2} + 2$$

$$A = \left(\frac{5\pi}{2} + 2\right)r^2$$

$$h = \frac{1}{2r}\left[\left(\frac{5\pi}{2} + 2\right)r^2\right]$$

$$= \frac{r}{2}\left(\frac{5\pi}{2} + 2\right) - \frac{\pi r}{4}$$

$$= \frac{5\pi r}{4} + r - \frac{\pi r}{4}$$

$$= \pi r + r$$

$$= r(\pi + 1)$$

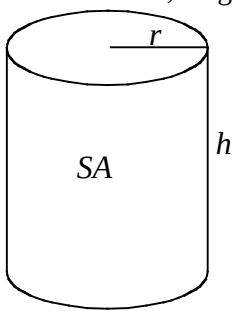
So the ratio is:

$$\frac{h}{r} = \frac{r(\pi + 1)}{r}$$

$$= \pi + 1$$

**Chapter 3 Section 6** **Question 8 Page 201**

**a)** Let the radius, height, and surface area be represented by $r$, $h$, and S.A. respectively.




S.A. is the variable to be maximized. Need to eliminate one of the other variables.
The volume is 1.

$$1 = \pi r^2 h$$

$$h = \frac{1}{\pi r^2}$$

The surface area is:

$$\begin{aligned}\text{S.A.} &= 2\pi r^2 + 2\pi r h \\ &= 2\pi r^2 + 2\pi r\left(\frac{1}{\pi r^2}\right) \\ &= 2\pi r^2 + \frac{2}{r} \\ &= 2\pi r^2 + 2r^{-1}\end{aligned}$$

For a minimum, find the critical points.

$$\begin{aligned}\text{S.A.}' &= 4\pi r - 2r^{-2} \\ 0 &= 4\pi r - 2r^{-2} \\ 0 &= 4\pi r^3 - 2 \\ r &= \sqrt[3]{\frac{1}{2\pi}}\end{aligned}$$

$r \approx 0.54$

To verify this is a minimum, test values that are close.

$$\text{S.A.}(0.54) = 5.54$$
$$\text{S.A.}(0.5) = 5.57$$
$$\text{S.A.}(0.6) = 5.60$$

$$\begin{aligned} h &= \frac{1}{\pi r^2} \\ &= \frac{\sqrt[3]{4\pi^2}}{\pi} \\ &= \sqrt[3]{\frac{4}{\pi}} \end{aligned}$$

The surface area is a minimum when $r = \sqrt[3]{\frac{1}{2\pi}}$ and $h = \sqrt[3]{\frac{4}{\pi}}$ or, in decimals, when $r \approx 0.54$ and $h \approx 1.08$.

**b)**
$$\begin{aligned} \frac{h}{d} &= \frac{h}{2r} \\ &= \frac{1.08}{2(0.54)} \\ &= \frac{1}{1} \end{aligned}$$

The ratio of height to diameter is 1:1.

**c)** No. The diameter of a pop can is determined by the size of the hand that will hold it, irrespective of volume and cost of materials. Regular pop cans have a 2:1 ratio. Other types of drink cans may have different ratios depending on the volume of their contents. Some mini pop cans have a ratio close to 1:1.

**Chapter 3 Section 6 Question 9 Page 201**

Let the radius, height, and total cost be represented by $r$, $h$, and $C$ respectively.
$C$ is the variable to be minimized. Need to eliminate one of the other variables.

The volume is 500 cm$^{2.}$

$$V = \pi r^2 h$$
$$500 = \pi r^2 h$$
$$h = \frac{500}{\pi r^2}$$

The cost function is:

$$C(r) = (0.4)\pi r^2 + (0.2)\left(\pi r^2 + 2\pi rh\right)$$
$$= (0.4)\pi r^2 + (0.2)\left(\pi r^2 + 2\pi r\left(\frac{500}{\pi r^2}\right)\right)$$
$$= 0.6\pi r^2 + \frac{200}{r}$$
$$= 0.6\pi r^2 + 200r^{-1}$$

For a minimum cost, find the critical points.

$$C'(r) = 1.2\pi r - 200r^{-2}$$
$$0 = 1.2\pi r - 200r^{-2}$$
$$0 = 1.2\pi r^3 - 200$$
$$r = \sqrt[3]{\frac{500}{3\pi}}$$
$$r \approx 3.76$$

To verify this is a minimum, test values that are close.

$$C(3.76) = 79.84$$
$$C(3.7) = 79.86$$
$$C(3.8) = 79.85$$

The minimum occurs when $r = \sqrt[3]{\frac{500}{3\pi}}$ cm and $h = \frac{5\sqrt[3]{36\pi^2}}{\pi}$ cm or, in decimals, when $r \approx 3.76$ cm and $h = 11.27$ cm.

**Chapter 3 Section 6** **Question 10 Page 201**

**a)** Let the radius, height, and volume be represented by *r*, *h*, and *V* respectively.

$$V = \pi r^2 h$$

The surface area is 1 m$^2$.

$$\text{S.A.} = \pi r^2 + 2\pi rh$$
$$1 = \pi r^2 + 2\pi rh$$
$$h = \frac{1 - \pi r^2}{2\pi r}$$

$$V = \pi r^2\left(\frac{1 - \pi r^2}{2\pi r}\right)$$
$$= \frac{r - \pi r^3}{2}$$

**b)** For maximum volume, find the critical points.

$$V'(r) = \frac{1}{2}\left(1 - 3\pi r^2\right)$$

$$0 = \frac{1}{2}\left(1 - 3\pi r^2\right)$$

$$r^2 = \frac{1}{3\pi}$$

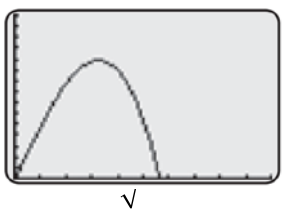
$$r = \pm\sqrt{\frac{1}{3\pi}}$$

$$r = \pm\frac{\sqrt{3\pi}}{3\pi}$$

$$r \approx \pm 0.33$$

Radius must be positive so discard –33 m. The maximum volume drum will occur when the radius is approximately 0.33 m.

**c)**



$(x \in [0, 1],\ \text{Xscl} = 0.1,\ y \in [0, 0.2],\ \text{Yscl} = 0.01)$

**d)** If the domain of $r$ is restricted to $0 < r < 0.2$, then there are no critical values to consider. The maximum volume can only occur at one of the endpoints. In fact, since this function is increasing on this interval (as is evident from the graph), the maximum volume will occur at the end of the interval, i.e., when $r = 0.2$.

$$V(0.2) = \frac{(0.2) - \pi(0.2)^3}{2}$$

$$= 0.1 - 0.004\pi$$

The maximum volume in this case will be 0.087 $m^3$.

**Chapter 3 Section 6** **Question 11 Page 202**

Let the radius, height, and volume of the paper cylinder be represented by $r$, $h$, and $V$ respectively.
$V$ is the variable to be maximized.

$$V = \pi r^2 h$$

Need this equation to only involve two variables.
The perimeter is to be 100 cm.

$$100 = 2h + 2(2\pi r)$$

$$r = \frac{50 - h}{2\pi}$$

Therefore, the formula becomes:

$$V(h)=\pi\left(\frac{50-h}{2\pi}\right)^2 h$$
$$=\frac{1}{4\pi}h^3-\frac{100}{4\pi}h^2+\frac{2500}{4\pi}h$$

For maximum volume, find the critical points.

$$V'(h)=\frac{3}{4\pi}h^2-\frac{200}{4\pi}h+\frac{2500}{4\pi}$$
$$0=\frac{3}{4\pi}h^2-\frac{200}{4\pi}h+\frac{2500}{4\pi}$$
$$0=3h^2-200h+2500$$
$$0=(3h-50)(h-50)$$
$$h=\frac{50}{3},\ h=50$$

Check the volume at the critical values.

$$V(50)=\frac{1}{4\pi}(50)^3-\frac{100}{4\pi}(50)^2+\frac{2500}{4\pi}(50)$$
$$=0$$
$$V\left(\frac{50}{3}\right)=\frac{1}{4\pi}\left(\frac{50}{3}\right)^3-\frac{100}{4\pi}\left(\frac{50}{3}\right)^2+\frac{2500}{4\pi}\left(\frac{50}{3}\right)$$
$$=1473.7$$

If one dimension of the paper is $\frac{50}{3}$, then the other dimension is $\frac{100-2\left(\frac{50}{3}\right)}{2}=\frac{100}{3}$

The maximum volume occurs when the paper is $\frac{50}{3}$ cm by $\frac{100}{3}$ cm.

**Chapter 3 Section 6 Question 12 Page 202**

Let $n$ be the number of additional trees planted and $P$ be the total crop.
The number of trees is now $50+n$ and the production per tree becomes $200-5n$.
$P=(50+n)(200-5n)$

To maximize the crop, find the critical points.

$P'(n)=(50+n)(-5)+(1)(200-5n)$ Product rule.

$$0=(50+n)(-5)+(1)(200-5n)$$
$$0=-250-5n+200-5n$$
$$n=-5$$

The optimal number of trees to be planted is 50 – 5 = 45.

Note: Increasing the number of trees above 50 actually reduces the total crop.

$P(45) = 10\ 125$

$P(50) = 10\ 000$

$P(55) = 9\ 625$

**Chapter 3 Section 6** **Question 13 Page 202**

**a)** $R(x) = (30 + x)(5000 - 100x)$

$= -100x^2 + 2000x + 150\ 000$

**b)** $-30 < x < 50$

Answers may vary. For example:
Assume that lowering the price increases the number of attendees. When $x = -30$, the ticket price is \$0, which is clearly impractical. If $x = 50$, then the number of attendees becomes zero which is also impractical.

**c)** Find the critical points.

$R'(x) = -200x + 2000$

$0 = -200x + 2000$

$x = 10$

To verify that this is a maximum, check values of $R(x)$..

$R(10) = 160\ 000$

$R(9) = 159\ 900$

$R(11) = 159\ 900$

A ticket price of \$40 will maximize the revenue from the concert.

**d)** The \$40 price results in 4000 people attending the concert. If only 1200 people can attend, the price should be set at \$68 (a \$38 price increase leads to a reduction of 3800 people). Any further increases in price will lead to a lower total revenue.

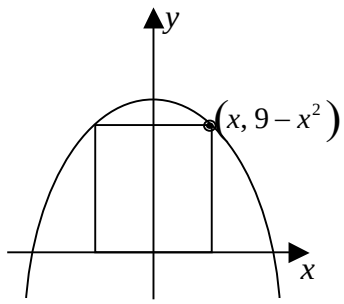
$R(38) = 81,\ 600$

$R(39) = 75\ 900$

$R(40) = 70\ 000$

**Chapter 3 Section 6** **Question 14 Page 202**

Let $(x, 9, -x^2)$ be any point on the parabola.
The resulting inscribed rectangle has width $2x$ and height $9 - x^2$.




Let $A$ represent the area.

$$\begin{aligned} A(x) &= 2x(9 - x^2) \\ &= 18x - 2x^3 \end{aligned}$$

For a maximum, find the critical points

$$\begin{aligned} A'(x) &= 18 - 6x^2 \\ 0 &= 18 - 6x^2 \\ x &= \sqrt{3} \end{aligned}$$

Check that this is a maximum.

$$A\left(\sqrt{3}\right) = 12\sqrt{3}$$

$$A(2) = 20$$

$$A(1) = 16$$

The rectangle with the largest area has area 20.8 units$^2$.

**Chapter 3 Section 6** **Question 15 Page 202**

**a)** To minimize fuel cost per kilometre, examine the critical points.

$$C(v) = \frac{v}{100} + \frac{25}{v}$$
$$= \frac{1}{100}v + 25v^{-1}$$
$$C'(v) = \frac{1}{100} - 25v^{-2}$$
$$0 = \frac{1}{100} - 25v^{-2}$$
$$0 = v^2 - 100(25)$$
$$v = \pm\sqrt{2500}$$
$$v = \pm 50$$

The negative value makes no sense for this problem. Check that $v$ = 50 is a minimum.

$$C(50) = 1.00$$
$$C(40) = 1.025$$
$$C(60) = 1.017$$

The speed that results in the lowest cost per kilometre is 50 km/h.

**b)** The total cost function, $T$, is made up from fuel cost and driver cost.

$$T(v) = \left(\frac{v}{100} + \frac{25}{v}\right)(1000) + 40\left(\frac{1000}{v}\right)$$
$$= 10v + 25\ 000v^{-1} + 40\ 000v^{-1}$$
$$= 10v + 65\ 000v^{-1}$$
$$T'(v) = 10 - 65\ 000v^{-2}$$
$$0 = 10 - 65\ 000v^{-2}$$
$$0 = 10v^2 - 65\ 000$$
$$v = \pm\sqrt{6500}$$
$$v \approx \pm 80.6$$

The negative value makes no sense for this problem. Check that $v$ = 80.6 is a minimum.

$$C(80) = 1612.50$$
$$C(80.6) = 1612.45$$
$$C(81) = 1612.47$$

The speed that results in the lowest total cost for the 1000-km trip is about 80.6 km/h

**Chapter 3 Section 6 Question 16 Page 202**

Let the number be $x$ and the sum to be minimized be $S$.

$$\begin{aligned} S(x) &= x^2 + \frac{1}{x} \\ &= x^2 + x^{-1} \end{aligned}$$

Examine the critical values.

$$\begin{aligned} S'(x) &= 2x - x^{-2} \\ 0 &= 2x - x^{-2} \\ 0 &= 2x^3 - 1 \\ x &= \sqrt[3]{0.5} \\ x &\approx 0.794 \end{aligned}$$

Check that $x = 0.794$ is a minimum.

$$\begin{aligned} S(0.794) &= 1.8899 \\ S(0.85) &= 1.899 \\ S(0.75) &= 1.8958 \end{aligned}$$

The number is $\sqrt[3]{0.5} \approx 0.794$.

**Chapter 3 Section 6 Question 17 Page 202**

Brenda's costs for the trip include the running costs for every 100 km (15 since she is travelling 1500 km) of the vehicle and the costs of paying herself.

$$\begin{aligned} C_{tot}(v) &= 15\left(0.9 + 0.0016v^2\right) + \frac{30(1500)}{v} \\ &= 13.5 + 0.024v^2 + \frac{45000}{v} \end{aligned}$$

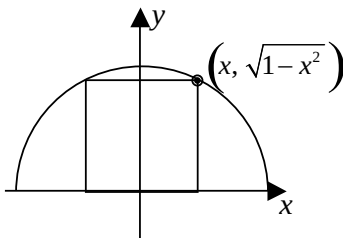
Find the derivative of the total cost and set it to zero to find the speed that will minimize cost.

$$\begin{aligned} C'_{tot}(v) &= 0.048v - \frac{45000}{v^2} \\ 0 &= 0.048v - \frac{45000}{v^2} \\ 0 &= 0.048v^3 - 45000 \\ 0.048v^3 &= 45000 \\ v^3 &= 937500 \\ v &\approx 97.9 \end{aligned}$$

The speed 97.9 km/h (approximately) will minimize Brenda's costs.

**Chapter 3 Section 6** **Question 18 Page 202**

Choose coordinate axes so that the plexiglass is a semicircle with equation $y=\sqrt{1-x^2}$ as shown.
Each point on the semicircle determines a possible rectangle.
The resulting inscribed rectangle has width $2x$ and height $\sqrt{1-x^2}$ .




Let $A$ represent the area.

$$\begin{aligned} A(x) &= 2x\sqrt{1-x^2} \\ &= 2x(1-x^2)^{\frac{1}{2}} \end{aligned}$$

For a maximum, find the critical points:

$$\begin{aligned} A'(x) &= 2x\left(\frac{1}{2}\right)(1-x^2)^{-\frac{1}{2}}(-2x)+2(1-x^2)^{\frac{1}{2}} \\ &= (1-x^2)^{-\frac{1}{2}}(-2x^2+2-2x^2) \\ &= (2-4x^2)(1-x^2)^{-\frac{1}{2}} \\ &= \frac{2-4x^2}{(1-x^2)^{\frac{1}{2}}} \\ 0 &= \frac{2-4x^2}{(1-x^2)^{\frac{1}{2}}} \\ x^2 &= \frac{1}{2} \\ x &= \pm\sqrt{0.5} \\ x &\approx \pm 0.71 \end{aligned}$$

The maximum point is confirmed by the graph of $A(x)$:.

QuickTime™ and a
TIFF (LZW) decompressor
are needed to see this picture.

The largest rectangle has dimensions 1.42 m by 0.71 m.

**Chapter 3 Section 6** **Question 19 Page 202**

**a)** If $x$ m are used for the quarter circle, then there are $(20 - x)$ m for the two sides of the square.

Each side will be $\left(\frac{20-x}{2}\right)$ m.

**b)** $4x = 2\pi r$

$$r = \frac{4x}{2\pi}$$

$$r = \frac{2x}{\pi}$$

**c)** $A(x) =$ area of square + area of quarter circle

$$= \left(\frac{20-x}{2}\right)^2 + \frac{1}{4}\pi\left(\frac{2x}{\pi}\right)^2$$

**d)** Simplifying,

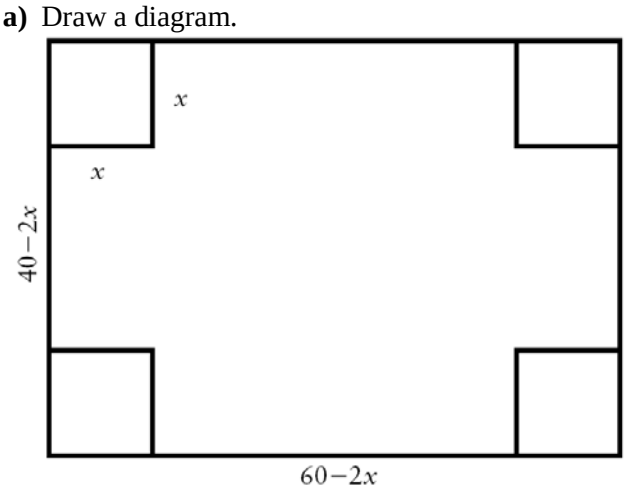
$$A(x) = \frac{400 - 40x + x^2}{4} + \frac{x^2}{\pi}$$

$$= 100 - 10x + \left(\frac{1}{4} + \frac{1}{\pi}\right)x^2$$

$$= \left(\frac{4+\pi}{4\pi}\right)x^2 - 10x + 100$$

**Chapter 3 Section 6** **Question 20 Page 202**

**a)** Draw a diagram.




**b)** $V(x) = x(40 - 2x)(60 - 2x)$

$$= 4x^3 - 200x^2 + 2400x$$

**c)** $0 < x < 20$

If $x < 0$, no tin would be cut from the corners and the box would have no volume.
If $x > 20$, all of the tin is cut away from the width and no box can be formed.

**d)** Find the critical values.

$$V'(x) = 12x^2 - 400x + 2400$$
$$0 = 12x^2 - 400x + 2400$$
$$0 = 3x^2 - 100x + 600$$
$$x = \frac{100 \pm \sqrt{10000 - 7200}}{6}$$
$$x = \frac{100 \pm 20\sqrt{7}}{6}$$
$$x = \frac{50 \pm 10\sqrt{7}}{3}$$
$$x \approx 25.5,\ x \approx 7.85$$

Check values for the one critical value in the domain for $x$.

$$V(7.85) = 8450.4$$
$$V(7) = 8372$$
$$V(9) = 8316$$

Therefore:

$$x = \frac{50 - 10\sqrt{7}}{3} \approx 7.85 \qquad 40 - 2x = \frac{20 + 20\sqrt{7}}{3} \approx 24.3 \qquad 60 - 2x = \frac{80 + 20\sqrt{7}}{3} \approx 44.3$$

The box dimensions that will maximize the volume are 7.85 cm by 24.3 cm by 44.3 cm.

**Chapter 3 Section 6** **Question 21 Page 203**

**a)** Find the volume and surface area.

$$V = \pi r^2(2r) = 2\pi r^3 \qquad \text{S.A.} = 2\pi r^2 + 2\pi r(2r) = 6\pi r^2$$

The profit function is

$$P(r) = 7(6\pi r^2) - 10(2\pi r^3).$$
$$= 42\pi r^2 - 20\pi r^3$$

To maximize the profit, find the critical values.

$$P'(r) = 84\pi r - 60\pi r^2$$
$$0 = 84\pi r - 60\pi r^2$$
$$0 = 12\pi r(7 - 5r)$$
$$r = 0,\ r = 1.4$$

Check values for this critical point.

$P(1.4) = 86.21$

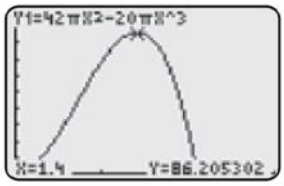
$P(1.5) = 84.82$

$P(1.3) = 84.95$

The container radius that maximizes the profit is 1.4 m.

**b)** The profit per container is $86.21

**c)**




**Chapter 3 Section 6** **Question 22 Page 203**

**a)** A cube is the most efficient shape. Non-cubical shapes require more material to enclose the same volume.

**b)** No. The volume involved does not affect the efficient shape considerations.

**c)** The shape would be expected to change.
Let the volume be 1 $m^3$. Also let the base be $x$ by $x$ and the height be $h$,

$$V = (x)(x)(h)$$
$$1 = (x)(x)(h)$$
$$h = \frac{1}{x^2}$$

The surface area is

$$\text{S.A.} = x^2 + 4x\left(\frac{1}{x^2}\right)$$
$$= x^2 + 4x^{-1}$$

For a minimum surface area, find the critical values.

$$\text{S.A.}' = 2x - 4x^{-2}$$
$$0 = 2x - 4x^{-2}$$
$$0 = 2x^3 - 4$$
$$x^3 = 2$$
$$x = \sqrt[3]{2}$$
$$x \approx 1.26$$

The corresponding height is:

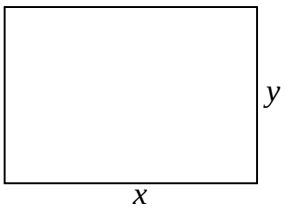
$$h = \frac{1}{\left(\sqrt[3]{2}\right)^2}$$

$$\approx 0.63$$

If the box has no top, the most efficient shape is a square base with a height that is one-half the base dimensions.

**Chapter 3 Section 6 Question 23 Page 203**

**a) i)** Let the dimensions be $x$ and $y$ as shown.




There is 1000 m of fencing.

$$2x + 2y = 1000$$
$$y = 500 - x$$

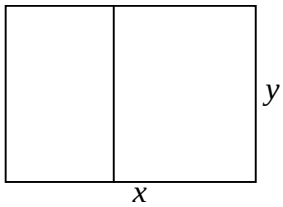
$$A(x) = x(500 - x)$$
$$= 500x - x^2$$

For a maximum:

$$A'(x) = 500 - 2x$$
$$0 = 500 - 2x$$
$$x = 250$$

The dimensions of the pen with maximum area are 250 m by 250 m.

**ii)**




There is 1000 m of fencing,

$$2x + 3y = 1000$$
$$y = \frac{1000 - 2x}{3}$$

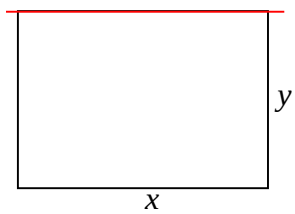
$$A(x) = x\left(\frac{1000 - 2x}{3}\right)$$
$$= \frac{1000}{3}x - \frac{2}{3}x^2$$

For a maximum:

$$A'(x) = \frac{1000}{3} - \frac{4}{3}x$$
$$0 = \frac{1000}{3} - \frac{4}{3}x$$
$$x = 250$$

The dimensions of the pen with maximum area are 250 m by 166.7 m.

**iii)**




There is 1000 m of fencing.

$$x + 2y = 1000$$
$$y = \frac{1000 - x}{2}$$

$$A(x) = x\left(\frac{1000 - x}{2}\right)$$
$$= 500x - \frac{1}{2}x^2$$

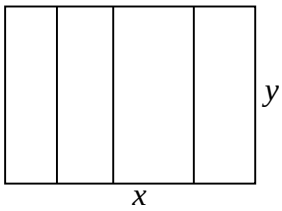
For a maximum,

$$A'(x) = 500 - x$$
$$0 = 500 - x$$
$$x = 500$$

The dimensions of the pen with maximum area are 500 m by 250 m.

**b)** The dimension with two sections of fence is always 250 m. The other dimension is 500 divided by the number of fence sections in that dimension.

**c)** The dimensions, if there were three dividers and four equal parts, would be 250 m by 100 m.
This result could be found by modifying the calculus technique used in part a).
It could also be found by using the pattern identified in part b).




**Chapter 3 Section 6** **Question 24 Page 203**

**a)** The cube has the least surface area. This can be verified using calculus, assuming the base is a square.
Let the volume be 1 $m^3$. Also let the base be $x$ by $x$ and the height be $h$,

$$V = (x)(x)(h)$$
$$1 = (x)(x)(h)$$
$$h = \frac{1}{x^2}$$

The surface area is

$$\text{S.A.} = 2x^2 + 4x\left(\frac{1}{x^2}\right)$$
$$= 2x^2 + 4x^{-1}$$

For a minimum surface area, find the critical values.

$$\text{S.A.}' = 4x - 4x^{-2}$$
$$0 = 4x - 4x^{-2}$$
$$0 = 4x^3 - 4$$
$$x^3 = 1$$
$$x = \sqrt[3]{1}$$
$$x = 1$$

The corresponding height is 1.
The dimensions are 1 m by 1 m by 1 m, a cube.

**b)** It appears that the best shape is one that has all dimensions equal. In three-space, that shape is a sphere. Compare a cube and a sphere. Let the volume be 1 $m^3$.

For the cube, the dimensions are 1 m by 1 m by 1 m and the total surface area is 6 $m^2$.

For the sphere:

$$1 = \frac{4}{3}\pi r^3$$

$$r = \sqrt[3]{\frac{3}{4\pi}}$$

$$r \approx 0.62$$

$$\text{S.A.} = 4\pi(0.62)^2$$

$$\approx 4.84$$

Clearly, the sphere has a smaller surface area than a cube of the same volume.

**c)** Answers may vary. For example:

:

- Packaging may not be a significant cost related to the product.
- Packaging is advertising. A larger surface pointed at the consumer provides the company with a l space in which to promote their product.
- Spheres are very hard to stack or store on rectangular shelves.
- Packages need to be handled. The size of the human hand may determine the thickness of the box.
- Aesthetically, a rectangle in the shape of the golden ratio is more pleasing than a square.

**Chapter 3 Section 6** **Question 25 Page 203**

The correct answer is E.

Let $r$ be the radius of the circle, where $0 \le r \le \frac{L}{2\pi}$.

The side length of the square is $\frac{L - 2\pi r}{4}$.

The total area is:

$$A = \pi r^2 + \left(\frac{L - 2\pi r}{4}\right)^2$$

$$\frac{dA}{dr} = \frac{\pi(\pi + 4)r}{2} - \frac{\pi L}{4}$$

$$\frac{dA}{dr} < 0 \text{ when } 0 \le r < \frac{L}{2(\pi + 4)}$$

$$\frac{dA}{dr} > 0 \text{ when } \frac{L}{2(\pi + 4)} < r \le \frac{L}{2\pi}.$$

Therefore, $A$ has a minimum when $r = \frac{L}{2(\pi + 4)}$.

Hence $2\pi r = \frac{\pi L}{\pi + 4}$ is used for the circle.

**Chapter 3 Section 6** **Question 26 Page 203**

The correct answer is E.

$$f(x)=\frac{x^n}{x-1}$$
$$f'(x)=\frac{x^{n-1}\left[(n-1)x-n\right]}{(x-1)^2}$$

If $n$ is even, then $f$ will have a local maximum at $x = 0$ and a local minimum at $x=\frac{n}{n-1}$.

If $n \neq 1$ is odd, then $f$ will have only a local minimum at $x=\frac{n}{n-1}$.

If $n = 1$, then $f$ will have no local maximum or minimum.

## Chapter 3 Review

### Chapter 3 Review Question 1 Page 204

**a)** $f(x) = 7 + 6x - x^2$

$f'(x) = 6 - 2x$

The function is increasing when $f'(x) > 0$. First solve the related equation.

$0 = 6 - 2x$

$x = 3$

Test the inequality for arbitrary values in the intervals (–∞, 3) and (3, ∞).
For the first interval, use $x = 0$.

$\text{L.S.} = 6 - 2(0)$ $\quad 4 > 0$

$= 4$

For the second interval, use $x = 4$.

$\text{L.S.} = 6 - 2(4)$ $\quad -2 < 0$

$= -2$

$f(x)$ is increasing on the interval (–∞, 3).
$f(x)$ is decreasing on the interval (3, ∞).

**b)** Let $y = f(x)$.

$f(x) = x^3 - 48x + 5$

$f'(x) = 3x^2 - 48$

The function is increasing when $f'(x) > 0$. First solve the related equation.

$3x^2 - 48 = 0$

$x^2 = 16$

$x = \pm 4$

Test the inequality for arbitrary values in the intervals (–∞, –4), (–4, 4), and (4, ∞).
For the first interval, use $x = -5$.

$\text{L.S.} = 3(-5)^2 - 48$ $\quad 27 > 0$

$= 27$

For the second interval, use $x = 0$.

$\text{L.S.} = 3(0)^2 - 48$ $\quad -48 < 0$

$= -48$

For the third interval, use $x = 5$.

$$\text{L.S.} = 3(5)^2 - 48 \qquad 27 > 0$$
$$= 27$$

$f(x)$ is increasing on the intervals $(-\infty, -4)$ and $(4, \infty)$.
$f(x)$ is decreasing on the interval $(-4, 4)$.

**c)** $g(x) = x^4 - 18x^2$

$g'(x) = 4x^3 - 36x$

The function is increasing when $g'(x) > 0$. First solve the related equation.

$$0 = 4x^3 - 36x$$
$$0 = 4x(x^2 - 9)$$

$$x = 0,\ x = \pm 3$$

Test the inequality for arbitrary values in the intervals $(-\infty, -3)$, $(-3, 0)$, $(0, 3)$, and $(3, \infty)$.
For the first interval, use $x = -5$.

$$\text{L.S.} = 4(-5)^3 - 36(-5) \qquad -320 < 0$$
$$= -320$$

For the second interval, use $x = -1$,

$$\text{L.S.} = 4(-1)^3 - 36(-1) \qquad 27 > 0$$
$$= 32$$

For the third interval, use $x = 1$.

$$\text{L.S.} = 4(1)^3 - 36(1) \qquad -32 < 0$$
$$= -32$$

For the fourth interval, use $x = 5$.

$$\text{L.S.} = 4(5)^3 - 36(5) \qquad 320 > 0$$
$$= 320$$

$g(x)$ is increasing on the intervals $(-3, 0)$ and $(3, \infty)$.
$g(x)$ is decreasing on the intervals $(-\infty, -3)$, $(0, 3)$.

**d)** $f(x) = x^3 + 10x - 9$

$f'(x) = 3x^2 + 10$

The function is increasing when $f'(x) > 0$. First solve the related equation.

$$0 = 3x^2 + 10$$
$$x^2 = -\frac{10}{3}$$

There are no solutions to this equation. Test the inequality for one arbitrary value, such as $x = 0$.

$$\text{L.S.} = 3(0)^2 + 10 \qquad 10 > 0$$
$$= 10$$

$f(x)$ is always increasing.

**Chapter 3 Review** **Question 2 Page 204**

$$f'(x) = x(x-3)^2$$
$$0 = x(x-3)^2$$

$x = 0$, $x = 3$

Test points in the intervals.

$$f'(-1) = -16$$
$$f'(1) = 4$$
$$f'(4) = 4$$

| | $x < 0$ | $x = 0$ | $0 < x < 3$ | $x = 3$ | $x > 3$ |
|---|---|---|---|---|---|
| $f'(x)$ | Negative | 0 | Positive | 0 | Positive |
| $f(x)$ | Decreasing | Minimum | Increasing | Point of inflection | Increasing |

**Chapter 3 Review** **Question 3 Page 204**

**a)** Let $y = f(x)$.

$$f(x) = 3x^2 + 24x - 8$$
$$f'(x) = 6x + 24$$
$$0 = 6x + 24$$
$$x = -4$$

Use a table to show increasing and decreasing intervals for the function.

| | $x < -4$ | $x = -4$ | $x > -4$ |
|---|---|---|---|
| Test value | –5 | –4 | 0 |
| $f'(x)$ | $f'(-5) = -6$ Negative | 0 | $f'(0) = 24$ Positive |
| y | Decreasing | (–4, –56) | Increasing |

The critical point (–4, –56) is a local minimum.

**b)** $f(x) = 16 - x^4$

$$f'(x) = -4x^3$$
$$0 = -4x^3$$
$$x = 0$$

Use a table to show increasing and decreasing intervals for the function.

| | $x < 0$ | $x = 0$ | $x > 0$ |
|---|---|---|---|
| Test value | –1 | 0 | 1 |
| $f'(x)$ | $f'(-1) = 4$ Positive | 0 | $f'(1) = -4$ Negative |
| $f(x)$ | Increasing | (0, 16) | Decreasing |

The critical point (0, 16) is a local maximum.

**c)** $g(x) = x^3 + 9x^2 - 21x - 12$

$$g'(x) = 3x^2 + 18x - 21$$
$$0 = 3x^2 + 18x - 21$$
$$0 = 3(x+7)(x-1)$$
$$x = -7,\ x = 1$$

Use a table to show increasing and decreasing intervals for the function.

| | $x < -7$ | $x = -7$ | $-7 < x < 1$ | $x = 1$ | $x > 1$ |
|---|---|---|---|---|---|
| Test value | –10 | –7 | 0 | 1 | 2 |
| $g'(x)$ | $g'(-10) = 99$ Positive | 0 | $g'(0) = -21$ Negative | 0 | $g'(2) = 27$ Positive |
| $g(x)$ | Increasing | (–2, 233) | Decreasing | (1, –23) | Increasing |

The critical point (–2, 233) is a local maximum and the point (1, –23) is a local minimum.

**Chapter 3 Review** **Question 4 Page 204**

**a)** Find the critical points for this function.

$$v(t) = 3t^2 - 24t + 88$$
$$v'(t) = 6t - 24$$
$$6t - 24 = 0$$
$$t = 4$$

Check that it is a minimum.

| | $x < 4$ | $x = 4$ | $x > 4$ |
|---|---|---|---|
| Test value | 3 | 4 | 5 |
| $v'(t)$ | $v'(3) = -6$<br>Negative | 0 | $v'(5) = 6$<br>Positive |
| $v(t)$ | Decreasing | (4, 40) | Increasing |

The critical point (4, 40) is a local minimum.
The minimum sped of the car is 40 km/h.

**b)** There are no critical points that can be maxima in this interval. The maximum must occur at an endpoint.

$v(2) = 52$

$v(5) = 43$

The maximum speed of the car in the interval is 52 km/h.

**Chapter 3 Review** **Question 5 Page 204**

Find the critical numbers.

$$f(x) = x^3 - 8x^2 + 5x + 2$$
$$f'(x) = 3x^2 - 16x + 5$$
$$0 = 3x^2 - 16x + 5$$
$$0 = (3x - 1)(x - 5)$$
$$x = \frac{1}{3}, x = 5$$

The critical numbers are $\frac{1}{3}$ and 5.

To find local extrema, use a table to show increasing and decreasing intervals for the function.

| | $x < \frac{1}{3}$ | $x = \frac{1}{3}$ | $\frac{1}{3} < x < 5$ | $x = 5$ | $x > 5$ |
|---|---|---|---|---|---|
| Test value | 0 | $\frac{1}{3}$ | 1 | 5 | 6 |
| $f'(x)$ | $f'(0) = 5$<br>Positive | 0 | $f'(1) = -8$<br>Negative | 0 | $f'(6) = 17$<br>Positive |
| $f(x)$ | Increasing | $\left(\frac{1}{3}, \frac{76}{27}\right)$ | Decreasing | (5, –48) | Increasing |

The critical point $\left(\frac{1}{3}, \frac{76}{27}\right)$ is a local maximum and the critical point (2, –48) is local minimum.

Check the function values at the endpoints of the interval.

$f(0) = 2$

$f(6) = -40$

The absolute minimum is (5, –48) and the absolute maximum is $\left(\frac{1}{3}, \frac{76}{27}\right)$ for the interval $[0, 6]$.

**Chapter 3 Review** **Question 6 Page 204**

B is the correct answer.
A cubic function will have a quadratic (second degree) first derivative and a linear (first degree) second derivative. A linear function always has exactly one zero and thus a cubic function must have exactly one point of inflection.

**Chapter 3 Review** **Question 7 Page 204**

True.
The second derivative of a quadratic function is of second degree. This derivative will have either zero, one, or two roots. Zero or two roots lead to zero or two points of inflection. If the quadratic has only one root, it means that it is always positive or always negative, which means that it does not change sign. In this case there can be no point of inflection because there is no sign change.

**Chapter 3 Review** **Question 8 Page 204**

$$f(x) = x^4 - 2x^3 - 12x^2 + 3$$
$$f'(x) = 4x^3 - 6x^2 - 24x$$
$$f''(x) = 12x^2 - 12x - 24$$

Find the possible points of inflection.

$$0 = 12x^2 - 12x - 24$$
$$0 = x^2 - x - 2$$
$$0 = (x-2)(x+1)$$
$$x = 2,\ x = -1$$

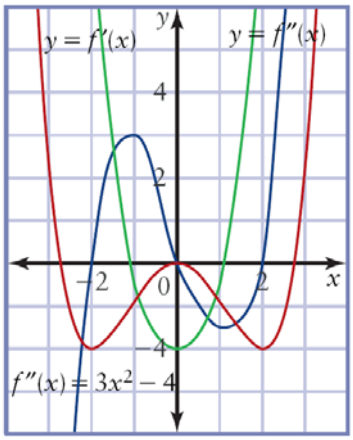
These values divide the domain into three intervals. Test the value of $f''(x)$ in each interval and summarize in a table.

| | $x < -1$ | $x = -1$ | $-1 < x < 2$ | $x = 2$ | $x > 2$ |
|---|---|---|---|---|---|
| Test value | –2 | –1 | 0 | 2 | 3 |
| $f''(x)$ | $f''(-2) = 48$ Positive | 0 | $f''(0) = -24$ Negative | 0 | $f''(3) = 48$ Positive |
| $f(x)$ | Concave up | Point of inflection (–1, –6) | Concave down | Point of Inflection (2, –45) | Concave up |

**Chapter 3 Review** **Question 9 Page 204**

**a)** From the graph of $f'(x)$, $f$ is increasing when $-2 < x < 0$ and $x > 2$ and deceasing otherwise. It will have local minima at $x = \pm 2$ and a local maximum at $x = 0$.

**b)** From the graph of $f'(x)$, $f''(x)$ will have zeros at –1.25 and +1.25.
$f''(x)$ will have negative values between –1.25 and +1.25 and positive values elsewhere. Since $f'(x)$ has an inflection point when $x = 0$, $f''(x)$ will have a local minimum at $x = 0$.




**Chapter 3 Review** **Question 10 Page 204**

Find the critical values.

$$f(x) = 2x^3 - x^4$$
$$f'(x) = 6x^2 - 4x^3$$
$$0 = 6x^2 - 4x^3$$
$$0 = 2x^2(3 - 2x)$$
$$x = 0,\ x = \frac{3}{2}$$

Use the second derivative test to determine maxima and minima.

$$f''(x) = 12x - 12x^2$$

Test the critical points.

$$f''\left(\frac{3}{2}\right) = -9$$

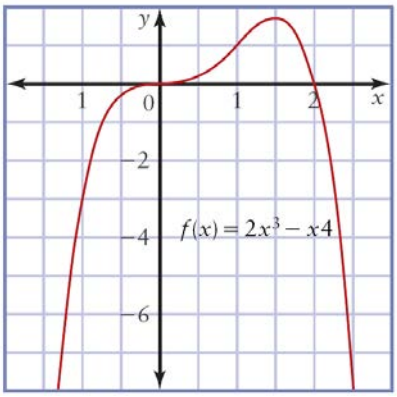
$$f''(0) = 0$$
$$f''(-0.5) = -9$$
$$f''(0.5) = 3$$

The second derivative test indicates that there is a local maximum at $x = 1.5$.
Since the sign of the second derivative changes at $x = 0$, there is a point of inflection there.

To sketch the curve, more information is needed.
Use a table to show increasing and decreasing intervals for the function.

| | $x < 0$ | $x = 0$ | $0 < x < 1.5$ | $x = 1.5$ | $x > 1.5$ |
|---|---|---|---|---|---|
| Test value | $-1$ | 0 | 1 | 0 | 2 |
| $f'(x)$ | $f'(-1) = 10$ Positive | 0 | $f'(1) = 2$ Positive | 0 | $f'(2) = -8$ Negative |
| $f(x)$ | Increasing | (0,0) | Increasing | (1.5, 1.7) | Decreasing |

Then sketch the graph.




**Chapter 3 Review** **Question 11 Page 204**

Vertical asymptotes only occur where the denominator in a rational function equals zero.

**a)** $x = 0$

**b)** $0 = 2x - 4$
$x = 2$

**c)** $0 = x^2 - 3x - 10$
$0 = (x - 5)(x + 2)$
$x = 5,\ x = -2$

**d)** $0 = x^2 + 2x + 1$
$0 = (x + 1)(x + 1)$
$x = -1$

**Chapter 3 Review** **Question 12 Page 205**

**a)** Substituting a small, positive number for $x$ means the numerator is close to 4 and the denominator is a very small, but positive number. The quotient of these two numbers will make a very large, positive number. Thus, $\lim_{x \to 0^+} f(x) = \infty$.

**b)** The limit has the same value, $\lim_{x \to 0^-} f(x) = \infty$, since substituting a small but negative number will still lead to a quotient of 4 and a small positive number.

**c)** Let $y = 0$.

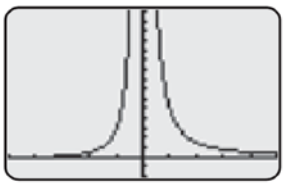
$$\frac{x+4}{x^2} = 0$$
$$x + 4 = 0$$
$$x = -4$$

**d)** To find the turning point, let the derivative equal zero.

$$f(x) = (x+4)x^{-2}$$
$$f'(x) = (x+4)(-2)x^{-3} + (1)x^{-2}$$
$$= x^{-3}(-2x - 8 + x)$$
$$0 = x^{-3}(-2x - 8 + x)$$
$$0 = -x - 8$$
$$x = -8$$

The turning point is $\left(-8, -\frac{1}{16}\right)$

**e)**



**Chapter 3 Review** **Question 13 Page 205**

**a)** $f(-x) = (-x)^3 - 3(-x)$
$$= -x^3 + 3x$$
$$= -f(x)$$
Therefore the function is an odd function.

**b)** This is a polynomial function. The domain is $\mathbb{R}$.

**c)** Let $x = 0$.
$$f(0) = (0)^3 - 3(0)$$
$$= 0$$
The $y$–intercept is 0.

Let $y = 0$.

$$0 = x^3 - 3x$$
$$0 = x(x^2 - 3)$$
$$x = 0,\ x = \pm\sqrt{3}$$

There are three $x$-intercepts at $-\sqrt{3}$, 0, and $\sqrt{3}$.

**d)** Find the critical points.

$$f(x) = x^3 - 3x$$
$$f'(x) = 3x^2 - 3$$
$$f''(x) = 6x$$
$$0 = 3x^2 - 3$$
$$0 = 3(x^2 - 1)$$
$$x = \pm 1$$

Use the second derivative test for extrema.

$$f''(-1) = 6(-1)$$
$$= -6$$
$$f''(1) = 6(1)$$
$$= 6$$

Therefore (1, –2) is a local minimum and (–1, 2) is a local maximum.

The critical points create three intervals. Test the derivative in each interval.

| | $x < -1$ | $-1 < x < 1$ | $x > 1$ |
|---|---|---|---|
| Test value | –2 | 0 | 2 |
| $f'(x)$ | $f'(-2) = 9$<br>Positive | $f'(0) = -3$<br>Negative | $f'(2) = 9$<br>Positive |
| $f(x)$ | Increasing | Decreasing | Increasing |

The function is increasing for $x < -1$ and $x > 1$ and is decreasing for $-1 < x < 1$.

For concavity, need to find all points of inflection. Let $f''(x) = 0$.

$$6x = 0$$
$$x = 0$$

Test for concavity in the intervals between the possible points of inflection.

| | $x < 0$ | $x = 0$ | $x > 0$ |
|---|---|---|---|
| Test value | $-1$ | | $1$ |
| $f''(x)$ | $f''(-1) = -6$ Negative | $0$ | $f''(1) = 6$ Positive |
| $f(x)$ | Concave down | Point of Inflection (0, 0) | Concave up |

**Chapter 3 Review** **Question 14 Page 205**

**a)** Follow the six-step plan.

$f(x) = -x^2 + 2x$

$f'(x) = -2x + 2$

$f''(x) = -2$

Step 1: Since this is a polynomial function, the domain is $\mathbb{R}$.

Step 2: $f(0) = 0$. The $y$-intercept is 0.

$0 = -x^2 + 2x$

$0 = x(x - 2)$

$x = 0, x = 2$

The $x$-intercepts are 0 and 2.

Step 3: Find the critical numbers.

$0 = -2x + 2$

$x = 1$

Use the second derivative test to classify the critical points.

$f''(1) = -2$

Therefore (1, 1) is a local maximum.

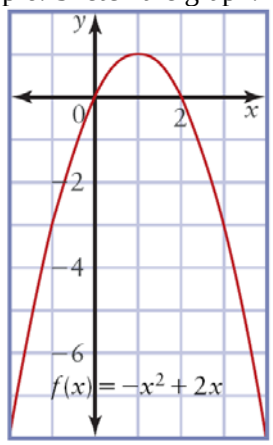
Step 4: Find the possible points of inflection.

Since $f''(x) - 2 \neq 0$ for any $x$, there are no points of inflection.

Step 5: From Step 3, $f(x)$ is decreasing for $x > 1$ and increasing for $x < 1$.

From Step 4, $f(x)$ is concave down for all $x$.

Step 6: Sketch the graph.




**b)** Follow the six-step plan.

$$k(x) = \frac{1}{4}x^4 - \frac{9}{2}x^2$$
$$k'(x) = x^3 - 9x$$
$$k''(x) = 3x^2 - 9$$

Step 1: Since this is a polynomial function, the domain is $\mathbb{R}$.

Step 2: $k(0) = 0$. The $y$-intercept is 0.

$$0 = \frac{1}{4}x^4 - \frac{9}{2}x^2$$
$$0 = x^2(x^2 - 18)$$
$$x = 0,\ x = \pm\sqrt{18}$$

The $x$-intercepts are 0 and $\pm 3\sqrt{2}$.

Step 3: Find the critical numbers.

$$0 = x^3 - 9x$$
$$0 = x(x^2 - 9)$$
$$x = 0,\ x = \pm 3$$

Use the second derivative test to classify the critical points.

$$k''(-3) = 18$$
$$k''(0) = -9$$
$$k''(3) = 18$$

Therefore (–3, –20.25) and (3, –20.25) are local minimum points and (0, 0) is a local maximum.

Step 4: Find the possible points of inflection.

$$0 = 3x^2 - 9$$
$$x^2 = 3$$
$$x = \pm\sqrt{3}$$

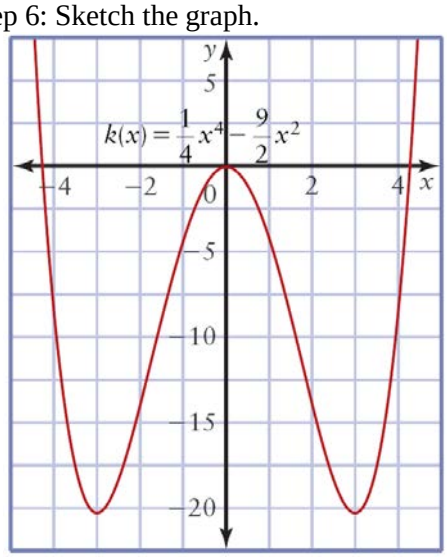
Test the intervals. Have already tested $x = 0, -3,$ and $3$.

Therefore $\left(-\sqrt{6}, -18\right)$ and $\left(\sqrt{6}, -18\right)$ are points of inflection.

Step 5: From Step 3, $k(x)$ is decreasing for $x < -3$ and $0 < x < 3$ and increasing for $-3 < x < 0$ and $x > 3$.

From Step 4, $k(x)$ is concave down for $-\sqrt{6} < x < \sqrt{6}$ and concave up elsewhere.

Step 6: Sketch the graph.




**c)** Follow the six-step plan.

$$h(x) = 2x^3 - 3x^2 - 3x + 2$$
$$h'(x) = 6x^2 - 6x - 3$$
$$h''(x) = 12x - 6$$

Step 1: Since this is a polynomial function, the domain is $\mathbb{R}$.

Step 2: $h(0) = 2$. The $y$-intercept is 2.

$$0 = 2x^3 - 3x^2 - 3x + 2$$
$$0 = 2(x^3 + 1) - 3x(x + 1)$$
$$0 = 2(x + 1)(x^2 - x + 1) - 3x(x + 1)$$
$$0 = (x + 1)(2x^2 - 2x + 2 - 3x)$$
$$0 = (x + 1)(2x^2 - 5x + 2)$$
$$0 = (x + 1)(2x - 1)(x - 2)$$
$$x = -1,\ x = \frac{1}{2},\ x = 2$$

The $x$-intercepts are –1, $\frac{1}{2}$ and 2.

Step 3: Find the critical numbers.

$$0 = 6x^2 - 6x - 3$$

$$0 = 3(2x^2 - 2x - 1)$$

$$x = \frac{2 \pm \sqrt{12}}{4}$$

$$x = \frac{1 \pm \sqrt{3}}{2}$$

$$x \approx -0.37, \ x \approx 1.37$$

Use the second derivative test to classify the critical points.

$$h''(-0.37) = -10.44$$

$$h''(1.37) = 10.44$$

Therefore (–0.37, 2.60) is a local maximum point and (1.37, –2.60) is a local minimum.

Step 4: Find the possible points of inflection.

$$0 = 12x - 6$$

$$x = \frac{1}{2}$$

Have already tested the intervals around 0.5.
Therefore (0.5, 0) is a point of inflection.

Step 5: From ③, $h(x)$ is decreasing for $\frac{1-\sqrt{3}}{2} < x < \frac{1+\sqrt{3}}{2}$ and increasing for $x < \frac{1-\sqrt{3}}{2}$ and $x > \frac{1+\sqrt{3}}{2}$.

From ④, $h(x)$ is concave down for $x < 0$ and concave up for $x > 0$.

Step 6: Sketch the graph.

QuickTime™ and a
TIFF (LZW) decompressor
are needed to see this picture.

**Chapter 3 Review** **Question 15 Page 205**

**a)** Find the critical points.

$$\begin{aligned}
C(t) &= \frac{0.12t}{t^2+2t+2} \\
&= 0.12t(t^2+2t+2)^{-1} \\
C'(t) &= 0.12t(-1)(t^2+2t+2)^{-2}(2t+2)+(0.12)(t^2+2t+2)^{-1} \\
&= (t^2+2t+2)^{-2}(-0.24t^2-0.24t+0.12t^2+0.24t+0.24) \\
&= \frac{-0.12t^2+0.24}{(t^2+2t+2)^2} \\
0 &= -0.12t^2+0.24 \\
0 &= -0.12(t^2-2) \\
t &= \pm\sqrt{2}
\end{aligned}$$

Since there is only one critical value in the interval $0 \le x \le 4$, test the function at the critical point and the endpoints.

$$\begin{aligned}
C(0) &= 0 \\
C\left(\sqrt{2}\right) &= 0.025 \\
C(4) &= 0.018
\end{aligned}$$

The maximum concentration of the drug during the interval $0 \le x \le 4$ is 0.025 mg/cm$^3$.

**b)** The maximum concentration occurs approximately 1.414 hours after administration of the drug.

**Chapter 3 Review** **Question 16 Page 205**

Use the variables suggested in the question.
Let S.A. be the surface area, the variable to be minimized.

$$\begin{aligned}\text{S.A.} &= 1(2w^2) + 2(2wh) + 2(wh)\\ &= 4w^2 + 6wh\end{aligned}$$

Volume is to be 2400 cm$^2$.

$$\begin{aligned}V &= 2w^2h\\ 2400 &= 2w^2h\\ h &= \frac{1200}{w^2}\end{aligned}$$

Simplify the surface area function.

$$\begin{aligned}\text{S.A.} &= 2w^2 + 6w\left(\frac{1200}{w^2}\right)\\ &= 2w^2 + 7200w^{-1}\end{aligned}$$

For a minimum, find the critical points.

$$\begin{aligned}\text{S.A.}' &= 4w - 7200w^{-2}\\ 0 &= 4w - 7200w^{-2}\\ 0 &= 4w^3 - 7200\\ w^3 &= 1800\\ w &= \sqrt[3]{1800}\\ w &\approx 12.16\end{aligned}$$

Test values on either side of this point.

$$\text{S.A.}'(11) = -15.5$$
$$\text{S.A.}'(13) = 9.40$$

Since the slope changes sign, the point must be a minimum.

$$\begin{aligned}2w &= 24.32\\ w &= 12.16\end{aligned}$$

$$\begin{aligned}h &= \frac{1200}{(12.16)^2}\\ &= 8.11\end{aligned}$$

The dimensions of the box will be 12.16 cm by 24.32 cm by 8.11 cm.

## Chapter 3 Practice Test

### Chapter 3 Practice Test Question 1 Page 206

B is the correct answer.

$$\begin{aligned} f'(x) &= 2x - 8 \\ 0 &= 2x - 8 \\ x &= 4 \\ f'(0) &= -8 \\ f'(3) &= -2 \end{aligned}$$

The only critical point is not in the given interval. The function is decreasing at both endpoints of the interval. Therefore it is always decreasing on the interval.

### Chapter 3 Practice Test Question 2 Page 206

D is the correct answer.

A, B, and C are incorrect. The function is decreasing for all values of *x*.

### Chapter 3 Practice Test Question 3 Page 206

C is the correct answer.

A: $(2, f(2))$ is a critical point because $f'(2) = 0$.
B: $(2, f(2))$ $(2, f(2))$ might be a turning point.
D: $(2, f(2))$ $(2, f(2))$ could be a local maximum (or a point of inflection) since *f* is increasing on the interval preceding $x = 2$. It certainly cannot be a local minimum.

### Chapter 3 Practice Test Question 4 Page 206

C is the correct answer.

For an odd function, $f(-x) = -f(x)$.
Therefore:

$$\begin{aligned} f(-a) &= -f(a) \\ &= -5 \end{aligned}$$

**Chapter 3 Practice Test** **Question 5 Page 206**

C is the correct answer.

A: The graph has no $x$-intercepts since the numerator of the fraction cannot be zero.

B: $f(x) = -3(x-2)^{-2}$

$f'(x) = 6(x-2)^{-3}$

$f''(x) = -18(x-2)^{-4}$

$= \dfrac{-18}{(x-2)^4}$

The second derivative is always negative (the denominator is always positive). The curve is always concave down.

C: $f'(0) = \dfrac{6}{-8}$

$= -0.75$

This contradicts the statement $f'(x) > 0$ when $x < 2$.

D: If a number close to 2 is substituted for $x$, the fraction becomes –3 divided by a very small positive number, the result of which is a very large negative number.

**Chapter 3 Practice Test** **Question 6 Page 206**

Given $f'(x) = x(x-1)^2$, the graph of $f(x)$ has **two** critical points and **one** turning point.

$f'(x) = x(x-1)^2$

$0 = x(x-1)^2$

$x = 0,\ x = 1$

Check the intervals.

$f'(-1) = -4$

$f'(0.5) = 0.125$

$f'(2) = 2$

There is a minimum when $x = 0$ and a point of inflection when $x = 1$.

**Chapter 3 Practice Test** **Question 7 Page 206**

A corresponds to a local maximum since slopes for $f(x)$ will change from positive to negative at that point.

B corresponds to a point of inflection since slopes for $f(x)$ will not change at that point but will be negative before and after the point.

C corresponds to a local minimum since slopes for $f(x)$ will change from negative to positive at that point.

**Chapter 3 Practice Test** **Question 8 Page 206**

$$f(x) = x^3 - 5x^2 + 6x + 2$$
$$f'(x) = 3x^2 - 10x + 6$$
$$0 = 3x^2 - 10x + 6$$
$$x = \frac{10 \pm \sqrt{28}}{6}$$
$$x = \frac{5 \pm \sqrt{7}}{3}$$
$$x \approx 2.5,\ x \approx 0.8$$

Check values at the critical points and the endpoints of the interval.

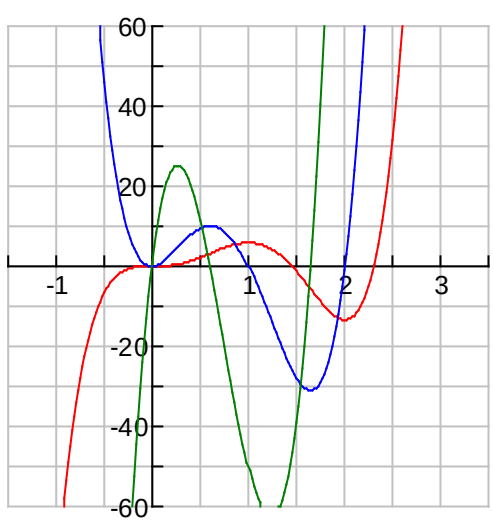
$$f(0) = 2$$
$$f\left(\frac{5-\sqrt{7}}{3}\right) \approx 4.1$$
$$f\left(\frac{5+\sqrt{7}}{3}\right) \approx 1.4$$
$$f(4) = 10$$

The absolute maximum is (4, 10) and the absolute minimum is (2.5, 1.4)

**Chapter 3 Practice Test** **Question 9 Page 206**

Answers may vary. For example:

$f(x)$ in red, $f'(x)$ in blue, and $f''(x)$ in green.
Note that zeros of $f'(x)$ correspond to local extrema on $f(x)$ and zeros of $f''(x)$ correspond to points of inflection on $f(x)$.



**Chapter 3 Practice Test** **Question 10 Page 207**

**a)** This is a quadratic polynomial function with leading coefficient +3. Would expect the limit to be $+\infty$ in both cases.

Substitute large numbers for $x$ to check.

$$f(100) = 284\ 180\ 000$$

$$f(-100) = 316\ 018\ 000$$

Clearly $\lim\limits_{x \to -\infty} f(x) = \infty$ and $\lim\limits_{x \to \infty} f(x) = \infty$.

**b)**

$$f(x) = 3x^4 - 16x^3 + 18x^2$$

$$f'(x) = 12x^3 - 48x^2 + 36x$$

$$f''(x) = 36x^2 - 96x + 36$$

Find the critical points.

$$0 = 12x^3 - 48x^2 + 36x$$

$$0 = 12x(x^2 - 4x + 3)$$

$$0 = 12x(x-3)(x-1)$$

$$x = 0,\ x = 1,\ x = 3$$

Test the critical points using the second derivative test.

$$\begin{aligned} f''(0) &= 36(0)^2 - 96(0) + 36 \\ &= 36 \end{aligned}$$

$$\begin{aligned} f''(1) &= 36(1)^2 - 96(1) + 36 \\ &= -24 \end{aligned}$$

$$\begin{aligned} f''(3) &= 36(3)^2 - 96(3) + 36 \\ &= 72 \end{aligned}$$

There are local minima at (0, 0) and (3, –27) and a local maximum at (1, 5).

**c)** For possible points of inflection, set the second derivative to zero.

$$0 = 36x^2 - 96x + 36$$

$$0 = 12\left(3x^2 - 8x + 3\right)$$

$$x = \frac{8 \pm \sqrt{64 - 36}}{6}$$

$$x = \frac{4 \pm \sqrt{7}}{3}$$

$$x \approx 0.45,\ x \approx 2.22$$

Test values for the intervals created by these points have already been checked above.

There are two points of inflection at (0.45, 2.31) and (2.22, –13.48).

**Chapter 3 Practice Test** **Question 11 Page 207**

**a)** $C(x) = 0.1x^2 + 1.2x + 3.6$

$$U(x) = \frac{0.1x^2 + 1.2x + 3.6}{x}$$
$$= 0.1x + 1.2 + 3.6x^{-1}$$

**b)** To minimize unit cost, find the critical points for $U(x)$.

$$U'(x) = 0.1 - 3.6x^{-2}$$
$$0 = 0.1 - 3.6x^{-2}$$
$$0 = x^2 - 36$$
$$x = \pm 6$$

The negative value has no meaning in this situation. Check the values around $x = 6$ .

$$U'(5) = 0.1 - 3.6(5)^{-2}$$
$$= -0.044$$
$$U'(7) = 0.1 - 3.6(7)^{-2}$$
$$= 0.027$$

There is a minimum when $x$ = 6. The company should produce 6 ATVs per day to minimize the cost per unit.

**Chapter 3 Practice Test** **Question 12 Page 207**

**a)** Vertical asymptotes occur when the denominator in a rational function is zero. In this case, the vertical asymptote is $x$ = 0.

**b)** Let $y = f(x)$.

$$f(x) = x^2 + \frac{1}{x^2}$$
$$= x^2 + x^{-2}$$
$$f'(x) = 2x - 2x^{-3}$$
$$f''(x) = 2 + 6x^{-4}$$

Find the critical points.

$$0 = 2x - 2x^{-3}$$
$$0 = x^4 - 1$$
$$x = \pm 1$$

Test using the second derivative test.

$f''(-1) = 8$

$f''(1) = 8$

There are local minima at (1, 2) and (–1, 2).

**c)** To find intervals of concavity, let $f''(x) = 0$

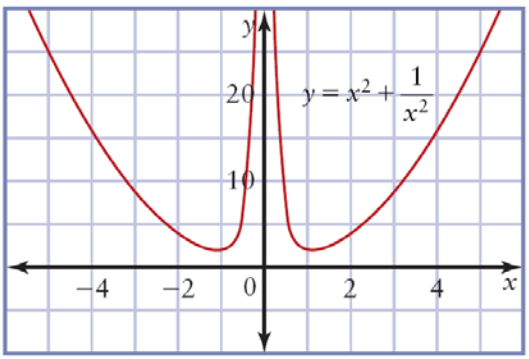
$0 = 2 + 6x^{-4}$

$0 = 2x^4 + 6$

$x^4 = -3$

There are no real roots to this equation.
The only value dividing regions of concavity is related to the asymptote.
Based on earlier tests, the function is concave up for $x < 0$ and also $x > 0$.

**d)**




**Chapter 3 Practice Test** **Question 13 Page 207**

**a) i)** $f'(0)$ B $-3$

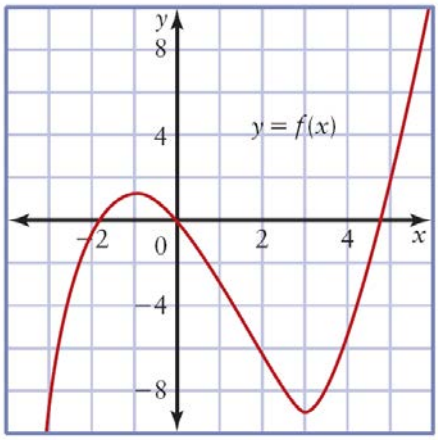
$f'(1)$ B $-4$

$f'(0) > f'(1)$

**ii)** Between $x = -1$ and $x = 3$, all the slopes for $f$ are negative (below zero on the $f'(x)$ graph).
Therefore, the function is decreasing from $x = -1$ to $x = 3$ and $f(-1) > f(3)$.

**iii)** There are no critical numbers between 5 and 10 and all the slopes of $f$ are positive in this interval.
Therefore, $f(10) > f(5)$

**b)** The sketch should have a maximum at $x = -1$ and, a minimum $x = 3$, and a point of inflection at x = 1.




**Chapter 3 Practice Test** **Question 14 Page 207**

Answers may vary. For example:

Since the factor in the denominator is squared, all of the values of the function will be positive. The curve will always be above the $x$-axis. Therefore, the limit as an asymptote is approached from either side must be $+\infty$

**Chapter 3 Practice Test** **Question 15 Page 207**

**a)** Let $x$ be the number of \$10 reductions in price. The revenue function is the number of rooms rented times the price per room.

$$\begin{aligned} R(x) &= (40+10x)(120-10x) \\ &= -100x^2 + 800x + 4800 \end{aligned}$$

$$\begin{aligned} R'(x) &= -200x + 800 \\ 0 &= -200x + 800 \\ x &= 4 \end{aligned}$$

To maximize revenue, the hotel should decrease the price by four increments to \$80 per room.

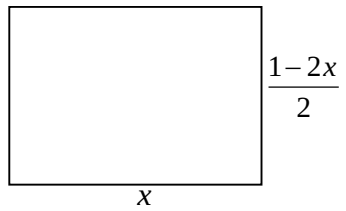
**b)** The revenue function is increasing until $x = 4$. If there are only 50 rooms, the hotel can only make one decrease of \$10. Compare the revenues.

$$R(0) = 4800$$
$$R(1) = 5500$$
$$R(4) = 6400$$

With 50 rooms, the maximum revenue is \$5500.

**Chapter 3 Practice Test** **Question 16 Page 207**

**a)**




Let the perimeter be 1 unit. The dimensions of the rectangle will be $x$ and $\frac{1-2x}{2}$.

$$\begin{aligned} A(x) &= x\left(\frac{1-2x}{2}\right) \\ &= 0.5x - x^2 \end{aligned}$$

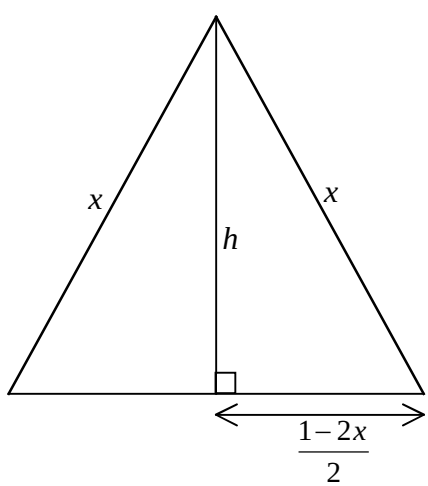
For a maximum area:

$$\begin{aligned} A'(x) &= 0.5 - 2x \\ 0 &= 0.5 - 2x \\ x &= 0.25 \end{aligned}$$

$$\frac{1-2(0.25)}{2} = 0.25$$

Therefore the dimensions are 0.25 units by 0.25 units.
The maximum area rectangle is a square.

**b)**




Assume that the triangle is isosceles.

Then for perimeter = 1,

$$x^2 = h^2 + \left(\frac{1-2x}{2}\right)^2$$

$$x^2 = h^2 + \frac{1}{4} - x + x^2$$

$$h = \sqrt{-\frac{1}{4} + x}$$

$$A(x) = \left(\frac{1}{2}\right)(1-2x)\sqrt{-\frac{1}{4}+x}$$

$$= (0.5-x)(x-0.25)^{\frac{1}{2}}$$

$$A'(x) = (\check{G}1)(x-0.25)^{\frac{1}{2}} + (0.5-x)\left(\frac{1}{2}\right)(x-0.25)^{-\frac{1}{2}}$$

For a maximum, let $A'(x) = 0$.

$$(\check{G}1)(x-0.25)^{\frac{1}{2}} + (0.5-x)\left(\frac{1}{2}\right)(x-0.25)^{-\frac{1}{2}} = 0$$

$$-(x-0.25) + (0.5-x)0.5 = 0$$ Multiply by $(x-0.25)^{\frac{1}{2}}$.

$$0.25 - x + 0.25 - 0.5x = 0$$

$$x = \frac{1}{3}$$

Therefore each side must be $\frac{1}{3}$ of the perimeter.

**c)** For a given perimeter, the octagon will enclose more area than the pentagon. The circle is the most efficient shape for a given perimeter in two dimensions and the octagon is closer to the circle shape than the pentagon would be. Again, the proof of these statements is beyond the scope of this course, but an investigation with specific numbers could confirm them.

**d)** In general, a circle or a square will enclose a maximum area for a perimeter.

**Chapter 3 Practice Test** **Question 17 Page 207**

**a)** For a maximum yield, let the derivative equal zero.

$$B(n) = -0.1n^2 + 10n$$

$$B'(n) = -0.2n + 10$$

$$0 = -0.2n + 10$$

$$n = 50$$

They should plant 50 000 seeds per acre for a maximum yield.

**b)** Create a profit function.

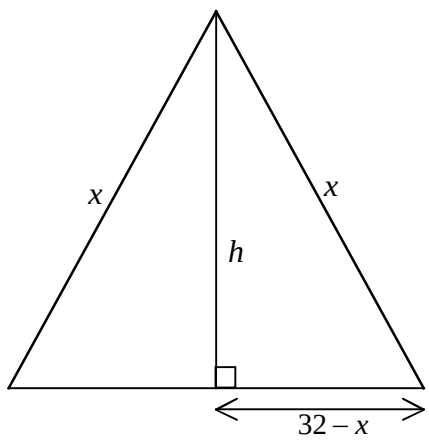
$$P(n) = 3(-0.1n^2 + 10n) - 2n$$
$$= -0.3n^2 + 28n$$

For a maximum profit, let the derivative equal zero.

$$P'(n) = -0.6n + 28$$
$$0 = -0.6n + 28$$
$$n = 46.667$$

For maximum profit, they should plant about 47 000 seeds per acre.

**Chapter 3 Practice Test** **Question 18 Page 207**




Let the two equal sides of the isosceles triangle have length $x$ cm.
In the right triangle:

$$x^2 = h^2 + (32 - x)^2$$
$$h = \sqrt{-1024 + 64x}$$

The area of the isosceles triangle is:

$$A(x) = \left(\frac{1}{2}\right)(64 - 2x)\sqrt{64x - 1024}$$
$$= (32 - x)\sqrt{64x - 1024}$$

For a maximum:

$$A'(x) = (-1)\sqrt{64x - 1024} + (32 - x)\left(\frac{1}{2}\right)(64x - 1024)^{-\frac{1}{2}}(64)$$

$$A'(x) = -\sqrt{64x - 1024} + (1024 - 32x)(64x - 1024)^{-\frac{1}{2}}$$

$$0 = -\sqrt{64x - 1024} + (1024 - 32x)(64x - 1024)^{-\frac{1}{2}}$$

$$0 = -64x + 1024 + 1024 - 32x$$

$$96x = 2048$$

$$x = \frac{64}{3}$$

$$x \approx 21.33$$

For a maximum area the three sides need to have the same length of approximately 21.33 cm.

## Chapter 1 to 3 Review

### Chapter 1 to 3 Review Question 1 Page 208

**a)** $\frac{h(20)-h(10)}{20-10}=\frac{0.04-0.82}{10}$

$\sqcap$ $\approx -0.078$

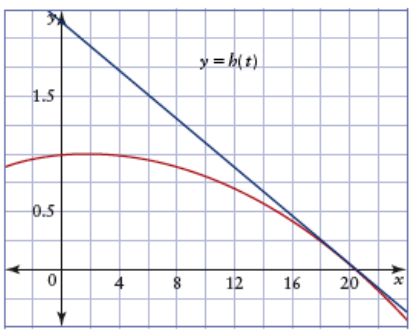
The average rate of change of the height is –0.078 m/s.
This is the average vertical velocity between 10 s and 20 s.

**b)** $\frac{h(20+0.1)-f(20)}{0.1}=\frac{0.02\,917-0.04}{0.1}$

$\sqcap$ $\approx -0.11$

The instantaneous rate of change at $t$ = 20 s is approximately –0.11 m/s.
This is the instantaneous vertical velocity at $t$ = 20 s.

**c)**




### Chapter 1 to 3 Review Question 2 Page 208

**a)** $t_1=\frac{3}{2};\hat{t}_2=\frac{6}{5};\hat{t}_3=\frac{9}{10};\hat{t}_4=\frac{12}{17};\hat{t}_5=\frac{15}{26}$

**b)** Yes, the limit is zero. As $x$ increases, the denominator increases faster than the numerator so the sequence approaches 0.

### Chapter 1 to 3 Review Question 3 Page 208

The limits as $x$ approaches 5 from the left and right sides are both equal to –2, so the limit as $x$ approaches 5 exists and equals –2. The value of the function at $x$ = 5 is not –2, so there is a removable discontinuity at this point. The function is discontinuous at $x$ = 5.

**Chapter 1 to 3 Review** **Question 4 Page 208**

**a)** $\lim_{x \to 3}(-2x^3 + x^2 - 4x + 7) = -50$

**b)** $\lim_{x \to 4} \frac{x^2 - 16}{x - 4} = \lim_{x \to 4} \frac{(x-4)(x+4)}{x-4}$

$= \lim_{x \to 4}(x+4)$

$= 8$

**c)** $\lim_{x \to 0} \frac{3x - 5}{7 + 4x^3} = -\frac{5}{7}$

**Chapter 1 to 3 Review** **Question 5 Page 208**

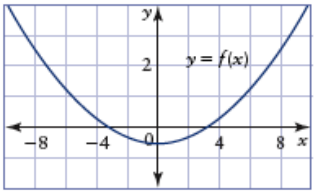
**a)** $\{x \mid x \neq 3, x \in \mathbb{R}\}$

**b)** $\lim_{x \to 3^+} f(x) = \infty$

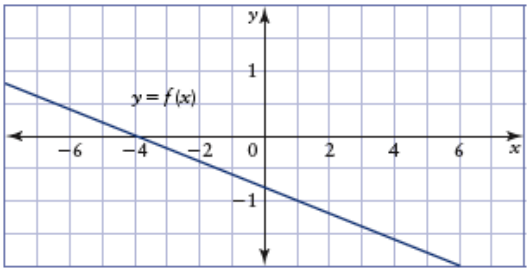
**c)** No. The graph is discontinuous at $x = 3$, since there is a vertical asymptote at this point.

**Chapter 1 to 3 Review** **Question 6 Page 208**

**a)**




**b)**



**Chapter 1 to 3 Review** **Question 7 Page 208**

**a)** $$\frac{dy}{dx} = \lim_{h\to 0}\frac{\frac{2}{3}(x+h)^3 - 8(x+h) - \left(\frac{2}{3}x^3 - 8x\right)}{h}$$
$$= \lim_{h\to 0}\frac{\frac{2}{3}(x^3 + 3x^2h + 3xh^2 + h^3) - 8(x+h) - \frac{2}{3}x^3 + 8x}{h}$$
$$= \lim_{h\to 0}\frac{2x^2h + 2xh^2 + \frac{2}{3}h^3 - 8h}{h}$$
$$= \lim_{h\to 0}(2x^2 + 2xh + \frac{2}{3}h^2 - 8)$$
$$= 2x^2 - 8$$

**b)** $$\frac{dy}{dx} = \lim_{h\to 0}\frac{\sqrt{2(x+h)-1} - \sqrt{2x-1}}{h}$$
$$= \lim_{h\to 0}\frac{\left(\sqrt{2(x+h)-1} - \sqrt{2x-1}\right)\left(\sqrt{2(x+h)-1} + \sqrt{2x-1}\right)}{h\left(\sqrt{2(x+h)-1} + \sqrt{2x-1}\right)}$$
$$= \lim_{h\to 0}\frac{2(x+h)-1-(2x-1)}{h\left(\sqrt{2(x+h)-1} + \sqrt{2x-1}\right)}$$
$$= \lim_{h\to 0}\frac{2}{\sqrt{2(x+h)-1} + \sqrt{2x-1}}$$
$$= \frac{2}{2\sqrt{2x-1}}$$
$$= \frac{1}{\sqrt{2x-1}}$$

**c)** $$\frac{dy}{dx} = \lim_{h\to 0}\frac{3(x+h)^2 - (x+h) + 1 - (3x^2 - x + 1)}{h}$$
$$= \lim_{h\to 0}\frac{3x^2 + 6xh + 3h^2 - x - h + 1 - 3x^2 + x - 1}{h}$$
$$= \lim_{h\to 0}\frac{6xh + 3h^2 - h}{h}$$
$$= \lim_{h\to 0}(6x + 3h - 1)$$
$$= 6x - 1$$

**d)** $$f'(t)=\lim_{h\to 0}\frac{\dfrac{-2(t+h)}{(t+h)+3}-\left(\dfrac{-2t}{t+3}\right)}{h}$$
$$=\lim_{h\to 0}\frac{\dfrac{(t+3)(-2t-2h)+2t(t+h+3)}{(t+h+3)(t+3)}}{h}$$
$$=\lim_{h\to 0}\frac{-6h}{h(t+h+3)(t+3)}$$
$$=\lim_{h\to 0}\frac{-6}{(t+h+3)(t+3)}$$
$$=-\frac{6}{(t+3)^2}$$

**Chapter 1 to 3 Review** **Question 8 Page 208**

**a)** $\dfrac{dy}{dx}=24x^3-15x^2+3$

**b)** $$\frac{dy}{dx}=1(4t-7)+(3+t)(4)$$
$$=8t+5$$

**c)** $$g'(x)=4(-2x^3+3)^3(-6x^2)$$
$$=-24x^2(-2x^3+3)^3$$

**d)** $$s'(t)=\frac{(5t-1)^{\frac{1}{2}}(1)-t\left(\dfrac{1}{2}\right)(5t-1)^{-\frac{1}{2}}(5)}{5t-1}$$
$$=\frac{1}{(5t-1)^{\frac{1}{2}}}-\frac{5t}{2(5t-1)^{\frac{3}{2}}}$$
$$=\frac{2(5t-1)-5t}{2(5t-1)^{\frac{3}{2}}}$$
$$=\frac{5t-2}{2(5t-1)^{\frac{3}{2}}}$$

**Chapter 1 to 3 Review** **Question 9 Page 208**

**a)** $$\begin{aligned}\frac{dy}{dx} &= \frac{(x+3)^2(-4)+4x(2)(x+3)}{(x+3)^4}\\ &= \frac{-4x^2-24x-36+8x^2+24x}{(x+3)^4}\\ &= \frac{4x^2-36}{(x+3)^4}\\ &= \frac{4(x+3)(x-3)}{(x+3)^4}\\ &= \frac{4(x-3)}{(x+3)^3}\end{aligned}$$

The slope of the tangent at $x = -2$ is $\left.\frac{dy}{dx}\right|_{x=-2}$

$$\begin{aligned}\left.\frac{dy}{dx}\right|_{x=-2} &= \frac{4(-2-3)}{(-2+3)^3}\\ &= -20\end{aligned}$$

When $x = -2$, $y = 8$.
Use the point $(-2, 8)$ and $m = -20$ to find $b$ in the equation of the tangent, $y = mx + b$.
$8 = -20(-2) + b$
$b = -32$

The equation of the tangent at $x = -2$ is $y = -20x - 32$.

**b)** The slope of the tangent at $x = -1$ is $\left.\frac{dy}{dx}\right|_{x=-1}$

$$\begin{aligned}\left.\frac{dy}{dx}\right|_{x=-1} &= \frac{4(-1-3)}{(-1+3)^3}\\ &= -2\end{aligned}$$

The normal is perpendicular to the tangent, so the slope of the normal is $-\left(\frac{1}{-2}\right) = \frac{1}{2}$.

When $x = -1$, $y = 1$.

Use the point (–1, 1) and $m = \frac{1}{2}$ to find $b$ in the equation of the normal, $y = mx + b$.

$$1 = \frac{1}{2}(-1) + b$$

$$b = \frac{3}{2}$$

The equation of the normal at $x = -1$ is $y = \frac{1}{2}x + \frac{3}{2}$.

**Chapter 1 to 3 Review** **Question 10 Page 208**

**a)** $y - x + 2 = 0$

$y = x - 2$

The slope of the tangent lines is perpendicular to the line above, so the tangents have a slope of –1.

Now, find $f'(x)$ and the values of $x$ where $f'(x) = -1$.

$$f'(x) = 3x^2 + 4x$$
$$-1 = 3x^2 + 4x$$
$$0 = 3x^2 + 4x + 1$$
$$0 = (3x + 1)(x + 1)$$
$$x = -\frac{1}{3}, \text{Ê} x = -1$$

Therefore, the points are $\left(-\frac{1}{3}, \text{Ê}\frac{5}{27}\right)$ and (–1, 1).

**b)** The slope of the tangents is –1.

Use the point $\left(-\frac{1}{3}, \text{Ê}\frac{5}{27}\right)$ and $m = -1$ to find $b$ in the equation of the first tangent, $y = mx + b$.

$$\frac{5}{27} = -\left(-\frac{1}{3}\right) + b$$

$$b = -\frac{4}{27}$$

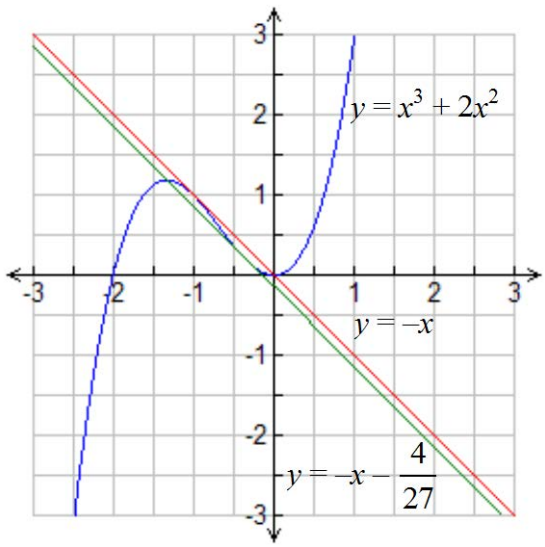
Use the point (–1, 1) and $m = -1$ to find $b$ in the equation of the second tangent, $y = mx + b$.

$1 = -1(-1) + b$

$b = 0$

Therefore, the equation of the first tangent is $y = -x - \frac{4}{27}$ and the equation of the second tangent is $y = -x$.

**c)** Sketch the graph of the function and the tangent lines.




**Chapter 1 to 3 Review** **Question 11 Page 209**

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$
$$= (5 - 4u + 3u^2)\left(\frac{1}{2}(2-x)^{-\frac{1}{2}}(-1)\right)$$
$$= \left(5 - 4\sqrt{2-x} + 3(2-x)\right)\left(\frac{-1}{2\sqrt{2-x}}\right)$$
$$= \left(11 - 4\sqrt{2-x} - 3x\right)\left(\frac{-1}{2\sqrt{2-x}}\right)$$

$$\left.\frac{dy}{dx}\right|_{x=-2} = \left(11 - 4\sqrt{2-(-2)} - 3(-2)\right)\left(\frac{-1}{2\sqrt{2-(-2)}}\right)$$
$$= 2.25$$

**Chapter 1 to 3 Review** **Question 12 Page 209**

**a) i)** Graph 1; Velocity and acceleration are derivatives of position.

**ii)** Graph 2; Velocity is the derivative of position, which is cubic, so the velocity function is quadratic.

**iii)** Graph 3; Acceleration is the derivative of velocity, which is quadratic, so the acceleration function is linear.

**b)** The object is slowing down when $v(t) \times a(t) < 0$, so over the intervals $x < 1$ and $2 < x < 3$.
The object is speeding up when $v(t) \times a(t) > 0$, so over the intervals $1 < x < 2$ and $x > 3$.

**c)** If the slope of the position function is moving towards zero then the object is slowing down. If the slope is moving away from zero then the object is speeding up.

**Chapter 1 to 3 Review** **Question 13 Page 209**

**a)** $V(t) = (3.72 - 0.02t^2)(13.00 + 0.21t)$
$= -0.0042t^3 - 0.26t^2 + 0.7812t + 48.36$

**b)** $V'(t) = -0.0126t^2 - 0.52t + 0.7812$

This is the change of voltage as time changes.

**c) i)** $V'(3) = \check{G}0.0126(3)^2\ \check{G}0.52(3) + 0.7812$
$= -0.8922$

The rate of change of voltage at $t = 3$ s is –0.8922 V/s.

**ii)** $I'(t) = -0.04t$
$I'(3) = -0.04(3)$
$= -0.12$

The rate of change of current at $t = 3$ s is –0.12 A/s.

**iii)** $R'(t) = 0.21$
$R'(3) = 0.21$

The rate of change of resistance at $t = 3$ s is 0.21 Ω/s.

**Chapter 1 to 3 Review** **Question 14 Page 209**

**a)** $C'(x) = -0.002x + 1.5$
$C'(300) = 0.90$

The marginal cost is \$0.90 per gadget.

**b)** $C(301) - C(300) = -0.001(301)^2 + 1.5(301) + 500 - (-0.001(300)^2 + 1.5(300) + 500)$
$= 0.899$

The actual cost of producing the 301st gadget is \$0.90.

**c)** $R(x) = xp(x)$

$= 4.5x - 0.1x^2$

$R'(x) = 4.5 - 0.2x$

$R'(300) = 4.5 - 0.2(300)$

$= -55.50$

The marginal revenue is –\$55.50 per gadget.

$P(x) = R(x) - C(x)$

$= -0.099x^2 + 3x - 500$

$P'(x) = -0.198x + 3$

$P'(300) = 3 - 0.198(300)$

$= -56.40$

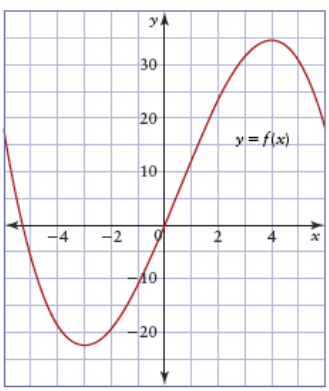
The marginal profit is –\$56.40 per gadget.

**Chapter 1 to 3 Review** **Question 15 Page 209**

**a) i)** $f'(2)$ since $f'(-2) = 6$ and $f'(2) = 10$.

**ii)** $f(6)$ since $f$ decreases from $x = 6$ to $x = 12$.

**b)**



**Chapter 1 to 3 Review** **Question 16 Page 209**

**a)** $f'(x) = 4x^3 - 12x^2$

$$0 = 4x^2(x-3)$$

$$x = 0, x = 3$$

When $x < 0$, $f'(x) < 0$.
When $0 < x < 3$, $f'(x) < 0$.
When $x > 3$, $f'(x) > 0$.

Therefore, (0, 0) is a point of inflection and (3, –27) is a local minimum.

**b)** $\frac{dy}{dx} = 6x^2 - 6x - 11$

$$0 = 6x^2 - 6x - 11$$

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(6)(-11)}}{2(6)}$$

$$x = \frac{3 \pm 5\sqrt{3}}{6}$$

$$x \approx 1.94, x \approx -0.94$$

When $x < -0.94$, $f'(x) > 0$.
When $-0.94 < x < 1.94$, $f'(x) < 0$.
When $x > 1.94$, $f'(x) > 0$.

Therefore, (–0.94, 12) is a local maximum and (1.94, –12) is a local minimum.

**c)** $h'(x) = 4x^3 - 15x^2 + 2x + 21$

$$0 = (x-3)(4x^2 - 3x - 7)$$

$$0 = (x-3)(4x-7)(x+1)$$

$$x = -1, x = \frac{7}{4}, x = 3$$

When $x < -1$, $f'(x) < 0$.

When $-1 < x < \frac{7}{4}$, $f'(x) > 0$.

When $\frac{7}{4} < x < 3$, $f'(x) < 0$.
When $x > 3$, $f'(x) > 0$.

Therefore, (–1, –32) is a local minimum, $\left(\frac{7}{4}, \frac{1125}{256}\right)$ is a local maximum, and (3, 0) is a local minimum.

**d)** $g'(x) = 12x^3 - 24x^2 - 12x + 24$

$0 = (x-1)(12x^2 - 12x - 24)$

$0 = 12(x-1)(x-2)(x+1)$

$x = -1, x = 1, x = 2$

When $x < -1$, $f'(x) < 0$.
When $-1 < x < 1$, $f'(x) > 0$.
When $1 < x < 2$, $f'(x) < 0$.
When $x > 2$, $f'(x) > 0$.

Therefore, (–1, –28) is a local minimum, (1, 4) is a local maximum, and (2, –1) is a local minimum.

**Chapter 1 to 3 Review** **Question 17 Page 209**

**a)** Follow the six-step plan.

$f(x) = x^3 - 9x^2 + 15x + 4$

$f'(x) = 3x^2 - 18x + 15$

$f''(x) = 6x - 18$

Step 1:Since this is a polynomial function, the domain is $\mathbb{R}$.

Step 2: $f(0) = 4$. The $y$-intercept is 4.
Since $x^3 - 9x^2 + 15x + 4 = 0$ is not factorable, do not look for $x$-intercepts at this time.

Step 3: Find the critical numbers.

$3x^2 - 18x + 15 = 0$

$3(x^2 - 6x + 5) = 0$

$3(x-5)(x-1) = 0$

$x = 5, x = 1$

Use the second derivative test to classify the critical points.

$f''(1) = -12$

$f''(5) = 12$

Therefore (5, –21) is a local minimum point and (1, 11) is a local maximum.

Step 4: Find the possible points of inflection.

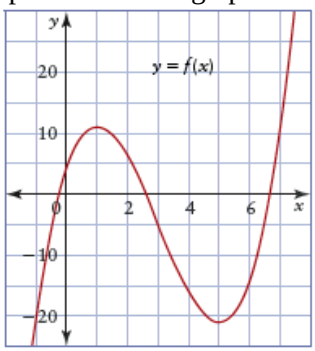
$6x - 18 = 0$

$x = 3$

Have already tested the intervals around $x = 3$.
Therefore (3, –5) is a point of inflection.

Step 5: From Step 3, $f$ is decreasing for $1 < x < 5$ and increasing for $x < 1$ and $x > 5$.
From Step 4, $f$ is concave down for $x < 3$ and concave up for $x > 3$.

Step 6: Sketch the graph.




**b)** Follow the six-step plan.

$f(x) = x^3 - 12x^2 + 36x + 5$

$f'(x) = 3x^2 - 24x + 36$

$f''(x) = 6x - 24$

Step 1: Since this is a polynomial function, the domain is $\mathbb{R}$.

Step 2: $f(0) = 5$. The $y$-intercept is 5.

Since $x^3 - 12x^2 + 36x + 5 = 0$ is not factorable, do not look for $x$-intercepts at this time.

Step 3: Find the critical numbers.

$3x^2 - 24x + 36 = 0$

$3(x^2 - 8x + 12) = 0$

$3(x - 6)(x - 2) = 0$

$x = 2, x = 6$

Use the second derivative test to classify the critical points.

$f''(2) = -12$

$f''(6) = 12$

Therefore (6, 5) is a local minimum point and (2, 37) is a local maximum.

④ Find the possible points of inflection.

$6x - 24 = 0$

$x = 4$

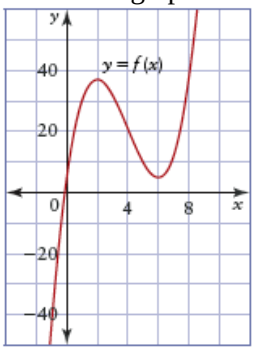
Have already tested the intervals around $x = 4$.

Therefore (4, 21) is a point of inflection.

⑤ From ③, $f$ is decreasing for $2 < x < 6$ and increasing for $x < 2$ and $x > 6$.

From ④, $f$ is concave down for $x < 4$ and concave up for $x > 4$.

⑥ Sketch the graph.




**c)** Follow the six-step plan.

$$f(x) = \frac{2x}{x^2 - 5x + 4}$$
$$= 2x(x^2 - 5x + 4)^{-1}$$

$$f'(x) = 2(x^2 - 5x + 4)^{-1} + 2x(-1)(x^2 - 5x + 4)^{-2}(2x - 5)$$
$$= (x^2 - 5x + 4)^{-2}(2x^2 - 10x + 8 - 4x^2 + 10x)$$
$$= (-2x^2 + 8)(x^2 - 5x + 4)^{-2}$$

$$f''(x) = (-4x)(x^2 - 5x + 4)^{-2} + (-2x^2 + 8)(-2)(x^2 - 5x + 4)^{-3}(2x - 5)$$
$$= (x^2 - 5x + 4)^{-3}(-4x^3 + 20x^2 - 16x + 8x^3 - 20x^2 - 32x + 80)$$
$$= (4x^3 - 48x + 80)(x^2 - 5x + 4)^{-3}$$

① Since this is a rational function, the domain is $\square$ except where there are vertical asymptotes.

$$x^2 - 5x + 4 = 0$$
$$(x - 4)(x - 1) = 0$$
$$x = 4,\ x = 1$$

There are vertical asymptotes at $x = 1$ and $x = 4$.

② $f(0) = 0$. The $y$-intercept is 0.
When $y = 0$, $x = 0$ is the only solution. The $x$-intercept is 0.

③ Find the critical numbers.

$$0 = (-2x^2 + 8)(x^2 - 5x + 4)^{-2}$$
$$0 = -2x^2 + 8$$
$$x^2 = 4$$
$$x = 2,\ x = -2$$

Use the second derivative test to classify the critical points.

$$f''(2) = -2$$
$$f''(-2) = 0.003$$

Therefore (–2, –0.22) is a local minimum point and (2, –2) is a local maximum.

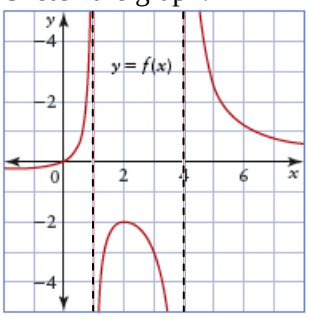
④ Find the possible points of inflection.

$$4x^3 - 48x + 80(x^2 - 5x + 4)^{-3} = 0$$
$$4x^3 - 48x + 80 = 0$$
$$4(x^3 - 12x + 20) = 0$$

This cubic equation does have a root but it is not readily factorable. There will be at least one point of inflection.

⑤ From ③, $f$ is decreasing for $2 < x < 4$ and increasing for $1 < x < 2$ .
From ④, $f$ is concave down for $1 < x < 4$ .

⑥ Sketch the graph.




**d)** Follow the six-step plan.

$$f(x) = 3x^3 + 7x^2 + 3x - 1$$
$$f'(x) = 9x^2 + 14x + 3$$
$$f''(x) = 18x + 14$$

① Since this is a polynomial function, the domain is $\mathbb{R}$.

② $f(0) = -1$. The $y$-intercept is –1.
For $x$-intercepts, let $y = 0$.

$$3x^3 + 7x^2 + 3x - 1 = 0$$

Try factoring using the factor theorem.
$f(1) = 12$
$f(-1) = 0$

Therefore, $(x + 1)$ is a factor.

$$(x+1)(3x^2 + 4x - 1) = 0$$

$x = -1$ and $x = \dfrac{-4 \pm \sqrt{28}}{6}$ or $x \approx -1.5$, $x \approx 0.2$

There are $x$-intercepts at –1.5, –1, and 0.2.

③ Find the critical numbers.

$0 = 9x^2 + 14x + 3$

$x = \dfrac{-14 \pm \sqrt{88}}{18}$

$x \approx -1.3,\ x \approx 0.3$

Use the second derivative test to classify the critical points.

$f''(-0.3) = 8.6$

$f''(-1.3) = -9.4$

Therefore (–0.3, –1.4) is a local minimum point and (–1.3, 0.4) is a local maximum.

④ Find the possible points of inflection.

$18x + 14 = 0$

$x = -\dfrac{7}{9}$

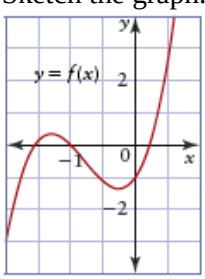
Have already tested the intervals around $x = -\dfrac{7}{9}$.

Therefore (–0.8, –0.5) is a point of inflection.

⑤ From ③, $f$ is decreasing for $-1.3 < x < -0.3$ and increasing for $x < -1.3$ and $x > -0.3$.

From ④, $f$ is concave down for $x < -\dfrac{7}{9}$ and concave up for $x > -\dfrac{7}{9}$.

⑥ Sketch the graph.




**e)** Follow the six-step plan.

$f(x) = x^4 - 5x^3 + x^2 + 21x - 18$

$f'(x) = 4x^3 - 15x^2 + 2x + 21$

$f''(x) = 12x^2 - 30x + 2$

① Since this is a polynomial function, the domain is $\mathbb{R}$.

② $f(0) = -18$. The $y$-intercept is –18.

For $x$-intercepts, let $y = 0$.

$x^4 - 5x^3 + x^2 + 21x - 18 = 0$

Try factoring using the factor theorem.
$f(1) = 0$

Therefore, $(x - 1)$ is a factor.

$$(x-1)(x^3 - 4x^2 - 3x + 18) = 0$$

Try factoring again using the factor theorem.
$f(3) = 0$

Therefore, $(x - 3)$ is a factor.

$$(x-1)(x-3)(x^2 - x - 6) = 0$$

$$(x-1)(x-3)(x-3)(x+2) = 0$$

$x = 1$, $x = 3$, $x = -2$

There are $x$-intercepts at –2, 1, and 3.

③ Find the critical numbers.

$$4x^3 - 15x^2 + 2x + 21 = 0$$

Use the factor theorem.
$f(-1) = 0$

Therefore, $(x + 1)$ is a factor.

$$(x+1)(4x^2 - 19x + 21) = 0$$

$$(x+1)(x-3)(4x-7) = 0$$

$x = -1$, $x = 3$, $x = 1.75$

Use the second derivative test to classify the critical points.

$$f''(-1) = 44$$

$$f''(3) = 20$$

$$f''(1.75) = -13.75$$

Therefore (–1, –32) and (3, 0) are local minimum points and (1.75, 4.4) is a local maximum.

④ Find the possible points of inflection.

$$0 = 12x^2 - 30x + 2$$

$$0 = 6x^2 - 15x + 1$$

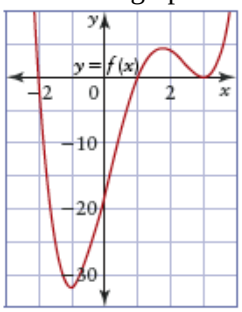
$$x = \frac{15 \pm \sqrt{201}}{12}$$

$x \approx 0.1$, $x \approx 2.4$

Have already tested the intervals around these values.
Therefore (0.1, –16.6) and (2.4, 2.1) are points of inflection.

⑤ From ③, $f$ is decreasing for $x < -1$ and $1.75 < x < 3$ and increasing for $1 < x < 1.75$ and $x > 3$.
From ④, $f$ is concave down for $0.1 < x < 2.4$ and concave up for $x < 0.1$ and $x > 2.4$.

⑥ Sketch the graph.




**Chapter 1 to 3 Review** **Question 18 Page 209**

The volume of a cylinder is $V = \pi r^2 h$

Use $V = 900$ to find an equation for $h$ in term of $r$.

$$900 = \pi r^2 h$$
$$h = \frac{900}{\pi r^2}$$

The surface area of a cylinder is S.A. = $2\pi r^2 + 2\pi rh$.

Substitute the equation for $h$ into S.A. to find the equation for the surface area in terms of $r$.

$$\text{S.A.} = 2\pi r^2 + 2\pi r\left(\frac{900}{\pi r^2}\right)$$
$$= 2\left(\pi r^2 + \frac{900}{r}\right)$$

The metal costs \$15.50 per m$^2$.

$$\frac{15.50}{(1)^2} = \frac{15.50}{(100)^2}$$
$$= 0.00155$$

The cost is \$0.00155/cm$^2$.

The function of cost is:

$$C(r) = 0.00155(\text{SA})$$
$$= 0.0031\left(\pi r^2 + \frac{900}{r}\right)$$

Find $C'(r)$ and set it to zero to find the radius at which cost will be optimized.

$$C'(r) = 0.0031\left(2\pi r - \frac{900}{r^2}\right)$$

$$0 = 2\pi r^3 - 900$$

$$r = \sqrt[3]{\frac{450}{\pi}}$$

The cost will be minimized when $r = \sqrt[3]{\frac{450}{\pi}}$ cm and $h = 2\sqrt[3]{\frac{450}{\pi}}$ cm.

$$C\left(\sqrt[3]{\frac{450}{\pi}}\right) = 0.0031\left(\pi\left(\sqrt[3]{\frac{450}{\pi}}\right)^2 + \frac{900}{\left(\sqrt[3]{\frac{450}{\pi}}\right)}\right)$$

$$\approx 0.0031(86.005 + 172.011)$$

$$\approx 0.80$$

The cost of making the can is $0.80.

**Chapter 1 to 3 Review** **Question 19 Page 209**

For the field to be rectangular, it must be split up into 4 equal rectangles side by side.
Let *l* represent the length of the whole field and *w* represent the width.
Let *P* represent the amount of fence needed.

$$P = 2l + 5w$$

$$6000 = 2l + 5w$$

$$l = \frac{6000 - 5w}{2}$$

The area of the whole field is:

$$A = l \times w$$

$$= \left(\frac{6000 - 5w}{2}\right) w$$

$$= 3000w - \frac{5}{2}w^2$$

Find $A'(w)$ and set it to zero to find the width of the field that will optimize area.

$$A'(w) = 3000 - 5w$$

$$0 = 3000 - 5w$$

$$w = 600$$

The width of the field is 600 m, which is also the width of each plot of land.

$$\frac{6000-5(600)}{2}=1500$$

The length of the whole field is 1500 m.

Since the field is split up into four congruent plots of land, each side length is 375 m.
Therefore, the dimensions of the land are 375 m by 600 m.